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p-adic periods and Perfectoid Spaces

NOTES: on Arizona Winter School web page

What are p-adic periods?

Classical periods are matrix entries for the comparison isoms among coho theories

$$\overbrace{H_j^B(X;\mathbb{Q}) \qquad\qquad H^j_{DR}(X)}^{\int_\gamma \omega}$$

$$\underset{\gamma}{\cup\!\!\!\!\!\!\;\;\;} \qquad\qquad\qquad \underset{\omega}{\in}$$

p-adic periods in this talk are what arise from comparison isomorphisms for algebraic varieties over finite extensions K of $\mathbb{Q}_p$. Replace Betti coh with étale coho.

Subtlety: periods not, in general, in

$$\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$$

e.g. X/K

$$V = H^i_{et}(X_{/\overline{K}}, \mathbb{Q}_p) \circlearrowleft G_K \quad \text{Gal rep.}$$

Then

$$\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p\right)^{G_K} \quad \text{is "too small"}$$

<u>The point</u> (Tate): $\mathbb{C}_p$ is the wrong ring here!

Instead, use

$$B_{HT} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$$

twist G_K action by powers of χ = cyclo char

"Hodge-Tate"

Then

$$D_{HT}(V) := \left(V \otimes_{\mathbb{Q}_p} B_{HT}\right)^{G_K}$$

has the right size:

$$D_{HT}(V) \otimes_K B_{HT} \cong V \otimes_{\mathbb{Q}_p} B_{HT}$$

Similar paradigm:

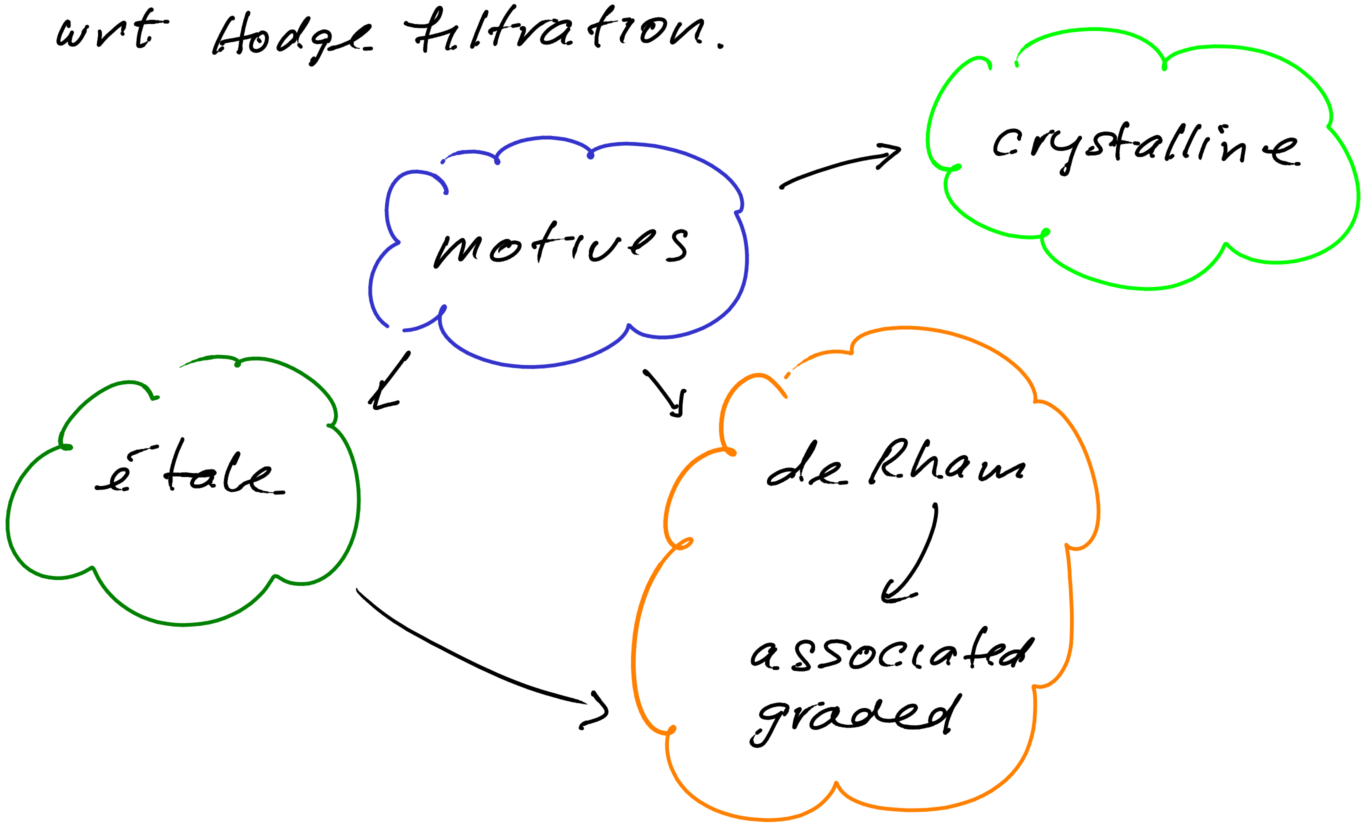
B_{*} = topological $\mathbb{Q}_p$-algebra with continuous G_K-action

$D_{*}(V) = (V \otimes_{\mathbb{Q}_p} B_{*})^{G_K}$ $\quad$ B_{*} – Graded mod

interesting extra structure

Want $D_{*}(V)$ to be some realiz^{n} of original motive.

eg: $D_{HT}(V)$ "is" associated graded of $H^i_{dR}(X)$ wrt Hodge filtration.



To lift étale $\to$ de Rham, replace B_{HT} by B_{dR} = a complete, discretely valued field with residue field $\mathbb{C}_p$ filtered by valuation

$$B_{cris} \hookrightarrow B_{dR}$$

φ = Frobenius

If X is smooth and proper over K with good reduction, then $D_{crys}(V)$ is big enough and

$$\mathcal{O}_K \to k$$

$$D_{cris}(V) \cong H^{i}_{cris}(X_k, \mathbb{Q}_p)$$

$$\uparrow$$

H^i_{dR} + Frobenius action

This is non canonically reversible

<u>Above</u>:

$$\overbrace{\text{Tate}, \ldots, \text{Fontaine}, \ldots \text{Faltings}}, \ldots, \text{Scholze}$$

new $\nearrow$

Perfectoid spaces: mutant rigid analytic spaces (adic spaces à la Huber) built out of base rings.

$$\mathbb{C}_p \langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty} \rangle$$

ie have $T_1, T_1^{1/p}, T_1^{1/p^2}, \dots$

This example can be found in work of Faltings. But this much more general.

kinda

Perfectoid spaces : rigid analytic spaces

as schemes : varieties.

For spaces and rings, there is a natural operation

$$X \rightsquigarrow X^\flat \qquad \text{"tilting"}$$

which preserves the underlying top space and the étale topology. But $X^\flat$ is in char p. It generalizes:

Fontaine–Wintenberger:

explicit isom

- $G_{\mathbb{Q}_p(\mu_{p^\infty})} \cong G_{\mathbb{F}_p((t))}$

- Faltings almost purity theorem: for certain finite étale morphisms

$$A[p^{-1}] \longrightarrow B[p^{-1}]$$

$A \to B$ is <u>almost</u> finite étale

eg

$$\begin{array}{c} \mathcal{O}_{\mathbb{C}_p}\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle \\ \downarrow \\ A \longrightarrow B \end{array}$$

This generalization of almost purity implies many deep conjectures of commutative algebra: $\begin{cases} \text{homological conjects} \\ \text{direct summand conjs.} \end{cases}$

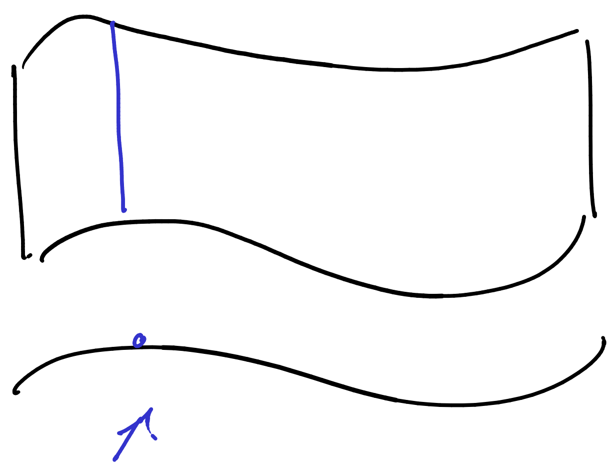
with some work

<u>Some applications to periods</u>

- Scholze: Étale – de Rham comparison remains valid for rigid analytic spaces (question of Tate). Also with coeffs: Kedlaya – Liu.

- Bhatt – Morrow – Scholze: establishes the "right" integral étale - crystalline comparison by constructing a realization between them.

- R. Liu - X. Zhu : under certain conditions, you can build compatible realizations without a motive:



family of étale realizns

on one fiber have DR realizn.

Then can construct compatible DR realization on all fibers.

This situation is common with Shimura varieties