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Galois Theory of Super Congruences

Periods:		rational #s:
relations between periods	$\longleftrightarrow$	prime power divisibility
Galois Theory	$\longleftrightarrow$	Galois Theory

Example:

$$\zeta(3) = \int_{[0,1]^3} \frac{dx\,dy\,dz}{1-xyz}$$

Approx of $\zeta(3)$

$$H_N(3) = \sum_{1}^{N} \frac{1}{n^3}$$

$$\zeta(3) = \frac{5}{2} \sum_{1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}$$

$$r_N = \frac{5}{2} \sum_{1}^{N} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}$$

Apery: a_N, b_N satis recursion:

$$N^3 u_N - (34N^2 - 51N^2 + 27N - 5)\, u_{N-1} + (N-1)^3 u_{N-2} = 0$$

$a_0 = 0$, $a_1 = 6$

$b_0 = 1$, $b_1 = 5$

$a_N / b_N \to \zeta(3) \notin \mathbb{Q}$

as convergence is fast enough.

<u>Def</u> $x, y \in \mathbb{Q}$. $x \equiv y \mod p^n$ if.

$\nu_p(x-y) \geq n$.

$$H_{p-1}(3) \equiv 0 \equiv a_{p-1} \mod p^2$$

$$H_{p-1}(3) \equiv \frac{3}{11} a_{p-1} \mod p^3$$

$$b_{p-1} \equiv 1 - 2p^3 r_{p-1} \mod p^5$$

$$b_{p-1} \equiv 1 - 2p^3 r_{p-1} + \frac{5}{9} p^3 H_{p-1}(3) + \frac{2}{3} p^3 a_{p-1} - 4/3\, p^6 r_{p-1} \mod p^8$$

<u>Def:</u> A <u>super congruence</u> is a congruence mod a prime power.

Should consider these in families, one for each prime.

$$(H_{p-1}), (r_{p-1}), (a_{p-1}), (b_{p-1}) \in \prod_p \mathbb{Q}_p$$

p-adic zeta functions

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \in \mathbb{Q}$$

For $n \in \mathbb{Z}$, $n \neq 1$.

Euler ϕ: $(p-1)p^{k-1}$

$$\zeta_p(n) := \lim_{k \to \infty} \zeta(-\phi(p^k)+n)$$

converges p-adically.

$\zeta_p(2n) = 0$

$\downarrow$

"p-adic analogue of π is 0"

$$H_{p-1}(1) = \sum_{1}^{p-1} \frac{1}{n}$$

PROP:

$$H_{p-1}(1) = -\sum_{n \geq 1} p^n \zeta_p(n+1) \in \mathbb{Q}_p.$$

Wolstenholme (1862)

$$H_{p-1}(1) \equiv 0 \mod p^2 \quad \leftrightsquigarrow \quad \zeta_p(2) = 0$$

More generally:

$$H_{p-1}(k) := \sum_{n=1}^{p-1} \frac{1}{n^k}$$

$$= (-1)^k \sum_{m=1}^{\infty} \binom{k+m-1}{k-1} p^m \zeta_p(k+m)$$

Multiple zeta values

$\underline{s} = (s_1, \ldots, s_k) \qquad s_j \in \mathbb{N}_{>0}$

MZV:

$$\zeta(\underline{s}) := \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \in \mathbb{R}$$

$s_1 > 1.$

MZVs are periods

p-adic MZVs $\zeta_p(s_1, \ldots, s_k) \in \mathbb{Q}_p$

Come from action of crystalline Frob on $\pi_1^{un}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})$. Expression in terms of iterated Coleman integrals.

Expect to satisfy same relns as MZV's along with $\zeta_p(2) = 0$. (They do satisfy all "motivic" relations.)

Remark:

$$\zeta_p(s_1, \ldots, s_k) \in \frac{1}{(s_1 + \cdots + s_k)!} \mathbb{Z}_p$$

Multiple harmonic sums

$$H_N(s_1, \ldots, s_k) = \sum_{N \geq n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \in \mathbb{Q}$$

<u>Thm</u> (Jarossay)

$$H_{p-1}(s_1, \ldots, s_k)$$

$$= \sum_{i=0}^{k} \sum_{l_1, \ldots, l_i \geq 0} (-1)^{s_1 + \cdots + s_i} \binom{s_1 + l_1 - 1}{l_1} \cdots \binom{s_i + l_i - 1}{l_i}$$

$$p^{l_1 + \cdots + l_i} \zeta_p(s_1 + l_1, \ldots, s_i + l_i) \zeta_p(s_{i+1}, \ldots, s_k)$$

Q: Why care about mult harmonic sums?

A: they show up everywhere.

Ex: $$\binom{2p}{p} = \frac{(p+1)(p+2) \cdots (p+p)}{1 \cdot 2 \cdot \cdots \cdot p}$$

$$= 2\left(1 + \frac{p}{1}\right)\left(1 + \frac{p}{2}\right) \cdots \left(1 + \frac{p}{p-1}\right)$$

$$= 2 \sum_{n=0}^{\infty} p^n H_{p-1}(1, \ldots, 1)$$

$$\binom{2p}{p} = 2 - 4p^3 \zeta_p(3) - 12 p^5 \zeta_p(5) - 4p^6 \zeta_p(3)^2 + O(p^7)$$

Motivic Galois Group

$\mathcal{P}^{\text{formal}}$ = ring of formal periods

G = motivic Galois gp. of $MTM(\mathbb{Z})$

$\mathcal{H}$ = ring of motivic MZVs (comm $\mathbb{Q}$-alg, spanned by $\zeta^m(\underline{s})$)

$$\mathrm{per}_p : \mathcal{H} \to \mathbb{Q}_p$$

$$\zeta^m(\underline{s}) \mapsto \zeta_p(\underline{s})$$

not injective as it kills $\zeta^m(2)$.

$$\mathcal{A} := \mathcal{H}/\zeta^m(2)$$

$$\mathrm{per}_p : \mathcal{A} \to \mathbb{Q}_p.$$

Where do supercongruences live?

$$\mathbb{Q}_{p-\infty} := \frac{\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \text{ bdd below}\}}{\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \to \infty}$$

Filtration:

$$\mathrm{Fil}^n \mathbb{Q}_{p\to\infty} = \{(a_p) : \lim_{p\to\infty} v_p(a_p) \geq n\}$$

Have

(*) $\quad per_{p\to\infty} \mathcal{A}[\![T]\!] \longrightarrow \mathcal{Q}_{p\to\infty}$

$$\zeta^m(\underline{s}) \longmapsto (\zeta_p(\underline{s}))$$

$$T \longmapsto (p)$$

Extend by continuity.

Ex $\left(\binom{2p}{p}\right) \in \mathcal{Q}_{p\to\infty}$

is in image of $per_{p\to\infty}$

<u>Period Conjecture</u> $p\to\infty$: (*) is an embedding of filtered algebras

$$per^{-1}_{p\to\infty}(\mathrm{Fil}^n \mathcal{Q}_{p\to\infty}) = T^n \mathcal{A}[\![T]\!]$$

Rosen: Paper computing G-action