2017.03.31

MSRI
Periods Workshop

Tannakian categories in a nutshell

F = field of char 0

Γ = any group.

$Rep_F(\Gamma)$ = category of f.dim reps of Γ in F vect spaces.

Have faithful $\omega : Rep_F(\Gamma) \to Vec_F$.
"fiber functor"

Axiomatize (carefully) to get axioms of a neutral F-linear tannakian category.

Theorem (Tannaka duality: cf Deligne)
If $\mathcal{C}$ is an F-linear tannakian category & $\omega : \mathcal{C} \to Vec_F$ is a fiber functor, then $\mathcal{C}$ is equivalent to the category of representations of

$$\pi_1(\mathcal{C}, \omega) := Aut^{\otimes} \omega$$

This is an affine group / F. Equiv,

it is a proalgebraic group. (Inverse limit of affine algebraic groups.)

__FACT:__ Every affine F group is an extension

$$1 \to U \to G \to R \to 1$$

prounipotent   pro reductive.

Rep (R) = semi-simple objects of $\mathcal{C}$.

U controls extensions of these

__FACT:__ (Levi) $G \to R$ is split and any 2 splittings are conjugate by an element of U:

$$G \cong R \ltimes U \quad \text{(not canonically)}$$

__Examples__

① Γ discrete group

$\mathcal{C}$ = category of unipotent reps of Γ / F

$$\Gamma^{un}_{/F} := \pi_1(\mathcal{C}, \omega).$$

Concrete:

$$\Gamma = \langle x_0, x_1 \rangle \cong F_2$$

$$\theta : \langle x_0, x_1 \rangle \longrightarrow F\langle\langle X_0, X_1 \rangle\rangle$$
$$x_j \longmapsto e^{X_j}$$

$$\Gamma^{un} \cong \exp \mathbb{L}(X_0, X_1)^{\wedge}$$

$$\mathrm{Lie}\, \Gamma^{un} = \mathbb{L}(X_0, X_1)^{\wedge}.$$

② $MHS_{\mathbb{Q}}$ is tannakian.

$\omega : MHS_{\mathbb{Q}} \to Vec_{\mathbb{Q}}$ Betti realizn.

③ Francis Brown's category $\mathcal{H}$
(Basically MHS enriched with $\mathbb{Q}$-DR structure.)

<u>HOLY GRAIL</u>: There is a $\mathbb{Q}$-linear tannakian category of mixed motives (with the "correct" exts)

Known in 1 case:

ring of S-ints in a # field

(1) Mixed Tate motives / $\mathcal{O}_{K,S}$

(Voevodsky, Levine, Deligne-Goncharov)

eg:

$$\pi_1(MTM(\mathbb{Z}), \omega^{DR})$$

$$\cong \mathbb{G}_m \ltimes \exp \mathbb{L}(\underbrace{\sigma_3, \sigma_5, \sigma_7}_{\text{gens \underline{not} canonical}}, \sigma_9, \ldots)^{\wedge}$$

$\mathbb{G}_m$ acts on σ_{2m-1} by

$$t \cdot \sigma_{2m-1} = t^{2m-1} \sigma_{2m-1}$$

gens not canonical

<u>Remark</u>: Elliptic story gives "natural choice" of σ_{2m-1}'s.

<u>Weight</u> <u>filtrations</u>: Objects V of MHS and mixed motives have a nat. weight filtration $W_\bullet$:

$$0 \subseteq \cdots \subseteq W_{r-1} V \subseteq W_r V \subseteq \cdots \subseteq V$$

Simple objects are "pure" ... only one (non-trivial) weight.

So have central cochar

$$\chi : \mathbb{G}_m \longrightarrow R = \pi_1(\mathcal{G}^{ss})$$

Levi ...

$$\begin{tikzcd} 1 \arrow[r] & U \arrow[r] & \pi_1(\mathcal{G}) \arrow[r] & R \arrow[r] \arrow[l, bend right, "s"'] & 1 \\ & \tilde{\chi} := s \circ \chi & & \mathbb{G}_m \arrow[ul] \arrow[u, "\chi"'] \end{tikzcd}$$

!

s, $s \circ \chi$ unique up to conj by elets of U.

$\tilde{\chi}$ acts on $\underline{u} = \mathrm{Lie}\, U$

Exercise: $\mathbb{G}_m$ acts on $\underline{u}$ with negative weights.

Eg: $\pi_1(MTM)$ above ...

$$\begin{tikzcd} \mathbb{G}_m \ltimes U \arrow[r] & \mathbb{G}_m & t^{-2} \\ & \mathbb{G}_m \arrow[u] & t \arrow[u, mapsto] \end{tikzcd}$$

as $\mathbb{Q}(1)$ has wt -2.

V in $\mathcal{G}$ $\circlearrowright$ $\pi_1(\mathcal{G})$ $\tilde{x} \uparrow$ $\mathbb{G}_m$ — Each choice of $\tilde{x}$ splits weight filt of V. Natural... respected by morphisms

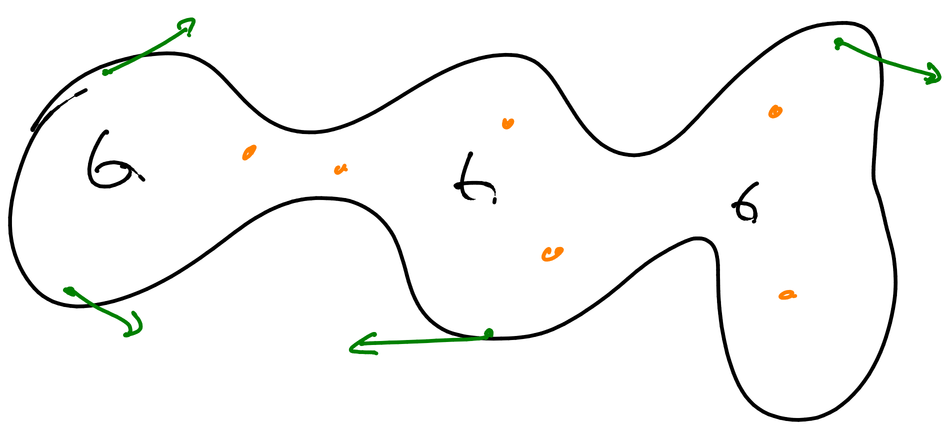
and gives natural isom

$$V \cong Gr_\bullet^W V$$

and implies that $Gr_\bullet^W$ is exact.

<u>Mapping class groups</u>:

S compact oriented surface genus g



with n (distinct) marked pts p_j
r non-zero tangent vects $\underline{v}_k$

anchored at distinct pt
(& none are at marked pt)

$\Gamma_{g,n+\vec{r}}$ = corresponding mapping class group

$3g-3+r+n \geq 0$

$= \pi_0 \, \mathrm{Diff}^+(S, \{P_1, \ldots, P_n\} \cup \{\vec{v}_1, \ldots, \vec{v}_r\})$

$\cong \pi_1(\mathcal{M}_{g,n+\vec{r}}, *)$

moduli of smooth genus g curves with n marked pt & r non-zero tgt vectors.

Eg: $\Gamma_{1,1} \cong SL_2(\mathbb{Z}) \cong \pi_1(\mathcal{M}_{1,1}, *)$

$\Gamma_{1,\vec{1}} \cong B_3$ = braid gp on 3-strings

$0 \to \mathbb{Z} \to \Gamma_{1,\vec{1}} \to \Gamma_{1,1} \to 1$

central extn: $\mathcal{M}_{1,\vec{1}} = \mathbb{A}^2 - \{u^3 - 27v^2 = 0\}$

Relative unipotent completion (of mapping class groups).

Have surjection

$$\begin{array}{rcl} \Gamma := \Gamma_{g,n+\vec{r}} & \to & \operatorname{Aut} H^{\circ}(S;\mathbb{Z}) \\ & & \wr\| \\ & & Sp(H^1(S,\mathbb{Z})) \\ & & \wr\| \\ & & Sp_g(\mathbb{Z}) \end{array}$$

Let $\mathcal{C}$ = cat of reps of Γ in finite dimensional $\mathbb{Q}$-vector spaces V that admit a filtration

$$0 = V_a \subseteq \cdots \subseteq V_{j-1} \subseteq V_j \subseteq \cdots \subseteq V_m = V$$

by Γ-submodules where Γ acts on each V_j / V_{j-1} via a representation of the algebraic group $Sp(H_{\mathbb{Q}}) \cong Sp_g(\mathbb{Q})$.

This is tannakian. Define

$$\mathcal{G}^{rel}_{g,n+\vec{r}} = \pi_1(\mathcal{C}, \omega)$$

$$1 \to \mathcal{U}^{rel}_{g,n+\vec r} \to \mathcal{G}^{rel}_{g,n+\vec r} \to Sp(H) \to 1$$

<u>Theorem</u>: For each choice of a base point

ie $\mathcal{O}(\mathcal{G}^{rel})$

$$x = (C, x_1, \dots, x_n, \vec v_1, \dots, \vec v_r)$$

This has a natural MHS. It has an F-DR structure if x is defined over F.

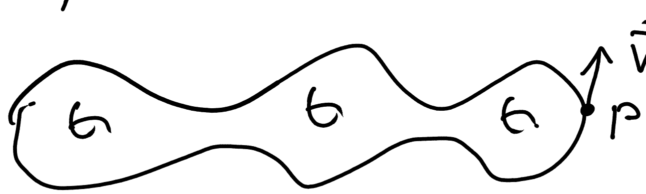
<u>Remark</u>: ① $\mathcal{G}_{g,n+\vec r} \otimes \mathbb{Q}_\ell$ has nice $G_{\mathbb{Q}}$ action when x defined $/\mathbb{Q}$.

② Later we'll take x to be a tangential base point of $\mathcal{M}_{g,n+\vec r}$. Then we have a limit MHS on $\mathcal{G}^{rel}_{g,n+\vec r}$.

<u>The geometric monodromy representation</u>

For $(C, \vec v) \in \mathcal{M}_{g,\vec 1}$ have

(1) $\mathfrak{p} := \mathrm{Lie}\, \pi_1^{un}(C', \vec v)$



$C' = C - \{P\}$

canonical isom

$$Gr^W_\bullet \mathfrak{p} \cong \mathbb{L}(H_1(C))$$

$$\circlearrowleft$$

$$Sp(H_{\mathbb{Q}})$$

(2) $\Gamma_{g,\vec{1}} = \pi_1(\mathcal{M}_{g,\vec{1}}, (C, \vec{v}))$ acts on $\mathfrak{p}$. This induces action

$$\mathcal{G}_{g,\vec{1}} \longrightarrow \operatorname{Aut} \mathfrak{p}$$

FACT: This is a morphism of MHS

Lie algebras: have

$$\mathfrak{g}_{g,\vec{1}} \longrightarrow \operatorname{Der} \mathfrak{p}$$

Pass to $Gr^W_\bullet$ to get

$$Gr\, \mathfrak{g}_{g,\vec{1}} \longrightarrow \operatorname{Der}^\theta \mathbb{L}(H)$$

not known to be injective

NO LOSS OF INFORMATION!

<u>What</u> <u>do</u> <u>we</u> <u>know</u> <u>about</u> $\mathfrak{g}_{g,\vec{1}}$?

(1) $Gr\, \mathfrak{g}_{g,\vec{1}} = \mathfrak{sp}(H) \ltimes Gr^W_\bullet \mathfrak{u}_{g,\vec{1}}$ ← negative weights

← canonical

(2) $g \geq 3$:

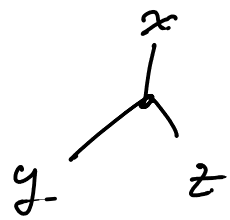
- $\mathrm{Gr}\, \underline{u}_{g,\vec{1}}$ is generated by $\Lambda^3 H$ (in weight -1)

Dennis Johnson + Hodge theory

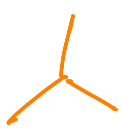
$$\mathrm{Gr}^W_{-1} \underline{u}_{g,\vec{1}} \longrightarrow \mathrm{Der}^\theta \mathbb{L}(H)$$

"Johnson homomorphism"

$$x \wedge y \wedge z \longmapsto$$



So geometric derivations are generated by



- $\mathrm{Gr}\, \underline{u}_{g,\vec{1}}$ is finitely presented
  - quadratic $g \geq 4$
  - cubic $g = 3$

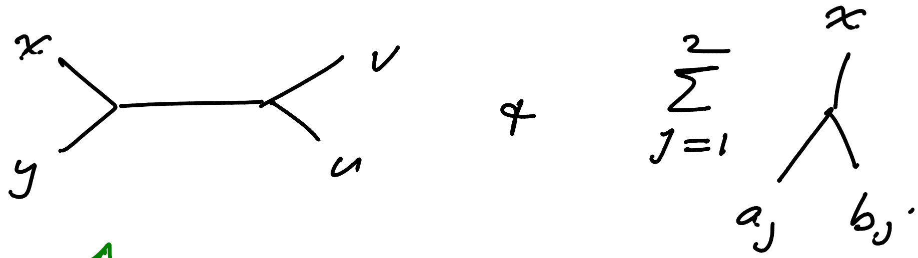
  ($\Rightarrow$ genus 3 MCGs not Kähler)

- $\mathrm{Gr}\, u_{2,\vec{1}}$ is fin presented

  Dan Petersen + Tatsunari Watanabe.

uses automorphic forms

$\mathrm{Gr}\, \underline{u}_{2,\vec{1}}$ gen by



outer

"almost inner"

Relation arithmetic ... come from Eisenstein series of $SL_2(\mathbb{Z})$:

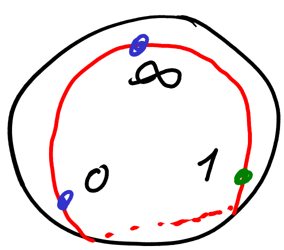
$$\underbrace{G_4, G_6, G_8, G_{10}, G_{14}}$$

weights in which there are no cusp forms!

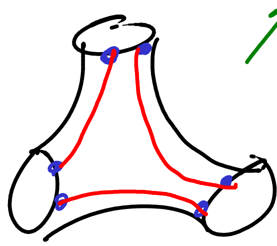
This leaves genus 1 (later). First...

Arithmetic monodromy

"Ihara curves": Basic pieces



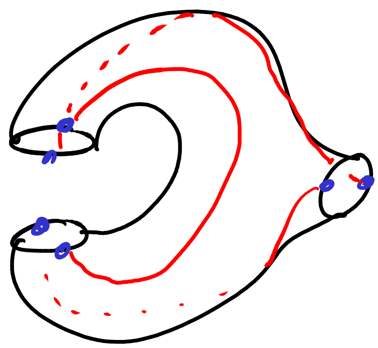
real oriented blow-up



$$= \widehat{Bl}_{\{0,1,\infty\}} \mathbb{P}^1$$

Assemble a surface of (say) type $(g, \vec{1})$ from these. (Blue pts must match.

__Ex__



"The punctured first order Tate curve" $E_{\partial/\partial q} - \{id\}$.

These correspond to integrally defined tangent vectors at 0-dim strata of $\overline{\mathcal{M}}_{g,1}$. Equiv: to first order smoothings of the stable curve obtained by contracting all boundary circles.

(Generalization of Tate curve due to Ihara & Nakamura.)

__Theorem:__ $\mathfrak{p} = \mathrm{Lie}\, \pi_1^{un}(C, \vec{v})$ is in $MTM(\mathbb{Z})$.

Corollary:

(1) $\mathrm{Der}\,\underline{p}$ is in $MTM(\mathbb{Z})$

(2) There is a homom

$$\underline{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \ldots) \to \mathrm{Der}^{\theta}\,\mathbb{L}(H).$$

? This depends on the pants decomp.

(3) (exercise) $\underline{k}$ normalizes the image of $\mathfrak{g}_{g,\vec{\imath}} \longrightarrow \mathrm{Der}\,\underline{p}$. Consequently, the image is in $MTM(\mathbb{Z})$.

(4) The image of $\underline{k}$ mod geom derivations is independent of the pants decomp.

(5) (Brown + Oda Conj)

$$\underline{k} \hookrightarrow \mathrm{Der}\,\underline{p} / \mathrm{im}\, \underline{u}_{g,\vec{\imath}}$$

FUNDAMENTAL PROBLEM:

Determine the image of

$$\underline{k} \longrightarrow \mathrm{Der}\, \mathfrak{p} / \mathrm{im}\, \underline{u}$$

Equivalently, determine the image of

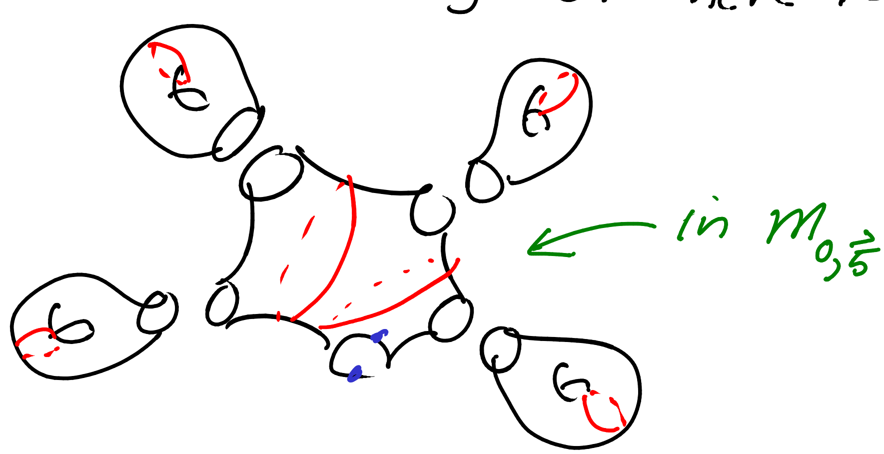
$$\underline{k} \ltimes \mathrm{im}\, u_{g,\vec{1}} \longrightarrow \mathrm{Der}\, \mathfrak{p}$$

Enough to do on $Gr^W_\bullet$: determine image of

$$\underline{k} \ltimes \mathrm{im}\, Gr\, \underline{u}_{g,\vec{1}} \longrightarrow \mathrm{Der}^\theta \mathbb{L}(H).$$

The most fundamental case is $g=1$.

The next most is $g=0$. Here is why:



The elliptic case:

(1) $\mathcal{G}_{1,\vec{1}} \cong \mathcal{G}_{1,1} \times \mathbb{Q}(1)$ ← trivial central extension.

So we have a monodromy rep

$$\mathcal{G}_{1,1} \longrightarrow \mathcal{G}_{1,\vec{1}} \longrightarrow \operatorname{Aut} \mathfrak{p}(E^\times_{,\vec{v}})$$

<u>Facts:</u>

① $\underline{u}_{1,1}$ is free $\cong \mathbb{L}(H_1(\underline{u}))^\wedge$

as $SL(H)$ mod

② $H_1(\underline{u}_{1,1})$

$$= \prod_{n>0} \Big[\overbrace{\bigoplus_f V_f \otimes S^{2n} H(2n+1)}^{\text{weight } -1} \oplus \overbrace{S^{2n} H(2n+1)}^{\text{weight } -2n-2} \Big]$$

equiv class → f eigen cusp form wt $2n+2$

↑ cuspidal part

↑ Eisenstein part.

as both MHS and $G_{\mathbb{Q}}$-module.

Here

(a) V_f = motive associated to f.

It is of weight $2n+1$ and Hodge type $(2n+1,0)$, $(0,2n+1)$.

(b) $H = H^1(E)$, where $E \in \mathcal{M}_{1,1}$ is the base point. When

$$E = E_{\partial/\partial q}, \quad H = \mathbb{Q} \oplus \mathbb{Q}(-1).$$

We'll use this base point.

Important remarks

(1) The motives (Hodge structures) that occur in $Gr^W \underline{u}_{1,1}$ include

$$S^{r_1} V_{f_1} \otimes \cdots \otimes S^{r_m} V_{f_m} (d)$$

$f_1, \ldots, f_m$ distinct (inequivalent) eigen cusp forms.

(2) These do not occur in

$$Gr_\cdot^W \operatorname{Der} \mathfrak{p} = \operatorname{Der} \mathbb{L}(\check{H})$$

$$\mathfrak{p} = \operatorname{Lie} \pi_1^{un}(E^\times_{\partial/\partial q})$$

$$H = H^1(E^\times_{\partial/\partial q}) = \mathbb{Q} \oplus \mathbb{Q}(-1)$$

$$\check{H} = H_1 = \mathbb{Q} \oplus \mathbb{Q}(1)$$

$\Rightarrow$ (3) the monodromy rep

$$\underline{u}_{1,1} \longrightarrow \operatorname{Der} \mathfrak{p}$$

has a large kernel. It contains ideal generated by the cuspidal generators.

(4) The image is in $MTM(\mathbb{Z})$.

The Eisenstein Quotient

Let $\mathfrak{g}^{eis}$ be the (unique) maximal quotient on which $\pi_1(MTM(\mathbb{Z}))$ acts.

$$\mathfrak{g}^{rel} \longrightarrow \mathfrak{g}^{eis}$$

$$\pi_1(MHS) \to \pi_1(MTM)$$

$$\mathfrak{g}^{eis} \cong \mathfrak{sl}(H) \ltimes \underline{u}^{eis}$$

$$\operatorname{Gr} \underline{u}^{eis} = \text{quotient of } \mathbb{L}\left(\bigoplus S^{2n} H(2n+1)\right)^{\wedge}$$

<u>NB</u>: $\pi_1(MTM(\mathbb{Z})) \circlearrowright \underline{\mathfrak{g}}^{eis}, \underline{u}^{eis}$

<u>Question</u>: Is $\underline{u}^{eis}$ free ?

NO! Each (normalized) Hecke eigenform gives a countable # of independent, minimal relations.

Better to seek relations in

$$Gr(\underline{k} \ltimes \underline{u}^{eis}) \circlearrowleft GL(H) \cong GL_2$$

so includes Galois action $\underline{k} \circlearrowleft \underline{u}^{eis}$

This is a quotient of

$$\mathbb{L}(\underbrace{z_3, z_5, \ldots,}_{} \bigoplus_{n\geq 1} S^{2n}(e_{2n+2}))$$

SL(H) acts trivially on these; z_{2m-1} has W wt = M wt $= -4m-2$.

copy of $S^{2n}H$ with highest weight vector e_{2n+2} of W-wt $-2n-2$ and Tate weight -2

Pollack relations

2. kinds of relations:

① (geometric) these are relations between the e_{2n}'s. These correspond to cusp forms.

② (arithmetic) these are relations of the form

$$[z_{2m-1}, e_{2n}] = \text{expression in the } e_{2a}'\text{s}.$$

These correspond to G_{2m}

Eisenstein series

There may be more ... but should not be.

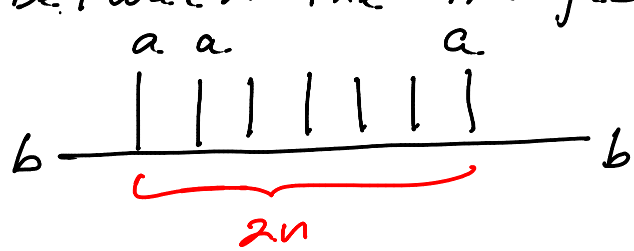
<u>History</u>:

① Predicted by Hain–Matsumoto (2007)

in $\mathrm{Der}^{\theta}\,\mathbb{L}(H)$

② Aaron Pollack found quad geom relations between the images

ε_{2n} =



in $\mathrm{Der}^{\theta}\mathbb{L}(H)$

of the ϵ_{2n}. And he found all other relations in $\mathrm{Der}^{\theta}\mathbb{L}(H)/\text{depth}\geq 3$. (2007 — 2009)

③ Using period computation of Francis Brown (of twice iterated integrals of Eisenstein series) Hain — Matsumoto showed they all lift from $\mathrm{Der}^{\theta}\mathbb{L}(H)/D^{\geq 3}$ to $\mathfrak{g}^{eis}_{1}$. We do not know them explicitly. Brown also has a proof.

<u>Remarks</u>:

① These relations are parametrized by

$$\underbrace{H^1(SL_2(\mathbb{Z}), S^{2n}H)^{\pm}}_{\substack{\mathbb{Q}\text{-vect space} \\ \text{type } (0,0) \\ \text{- trivial } G_{\mathbb{Q}}\text{-mod}}} \otimes S^{2n}H(d)$$

<u>Question</u>: Could they be related to occurrences of

$$H^1(SL_2(\mathbb{Z}), S^{2n}H)^{\pm} \otimes V_\lambda$$

in $H_1(\mathrm{Der}^{\theta}\,\mathbb{L}(H))$ in work of Conant – Kassabov – Vogtmann ?

(2) Pollack's relations appear to have remarkable arithmetic properties. He found congruences between the arithmetic relation associated to $691 \cdot G_{12}$ and Δ. (Similarly in wt 16).

The Infinitesimal Galois Action

How does $\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)$ act on $\mathbb{L}(H)$? We have section s

$$\begin{array}{c} \mathfrak{g}^{eis} \xrightarrow{\quad} \underline{k} \\ \parallel \\ \dfrac{\mathbb{L}(z_{2m-1},\ S^{2n}(e_{2n+2}))}{(\text{relations})} \end{array}$$

induced by $\partial/\partial q$ (base point)

This section is not $SL(H)$-invariant. It does not preserve the weight filtration $W_\bullet$, but does preserve $M_\bullet$, the relative weight filtration. We know

$$s : \sigma_{2m-1} \longmapsto \check{e}_{2m} + z_{2m-1} + \text{terms of "higher depth"}$$

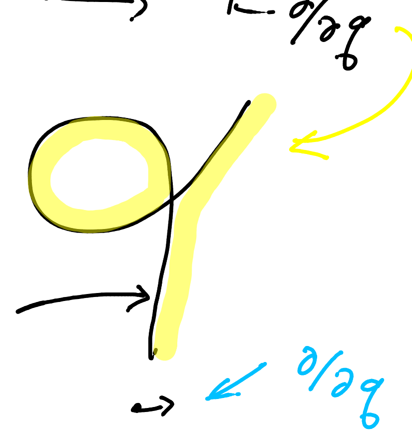
(Determining s is joint work with Francis Brown.)

<u>Relationship</u> <u>with</u> <u>genus</u> 0

Have "inclusion

$$\mathbb{P}^1 - \{0, 1, \infty\} \hookrightarrow E_{\partial/\partial q}$$

Picture 1:



$E_0 = \mathbb{P}^1/_{0 \sim \infty}$

$\mathring{E}_0 = \mathbb{G}_m$

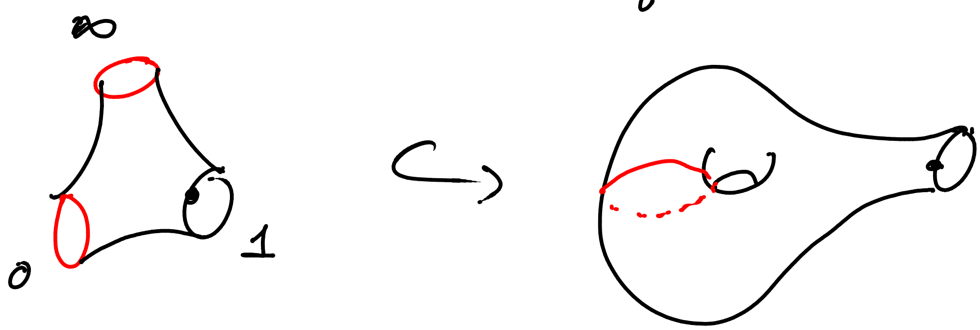
So
$$\begin{array}{ccc} \mathbb{P}^1 - \{0,1,\infty\} & \hookrightarrow & E^\times_{\partial/\partial q} \\ \| & & \\ \mathbb{G}_m - \{1\} & & \end{array}$$

Picture 2:

real oriented blow up

$$\begin{tikzcd} \mathbb{P}^1 - \{0,1,\infty\} \arrow[r, hook] & \widehat{Bl}_{\{0,1,\infty\}}\, \mathbb{P}^1 \arrow[d] \\ & E^\times_{\partial/\partial q} \end{tikzcd}$$

homotopy equiv



This induces

$$\pi_1^{un}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{v}) \to \pi_1^{un}(E^\times_{\partial/\partial q}, \vec{v})$$

This is $\pi_1(MTM(\mathbb{Z}))$-equivariant.

It therefore induces

$$\mathbb{L}(X_0, X_1)^\wedge \longrightarrow \mathbb{L}(A, T)^\wedge$$

FORMULA (Hain-Matsumoto, Enriquez)

$$X_1 \longmapsto \frac{T}{e^T - 1} \cdot A$$

$$X_0 \longmapsto [T, A]$$

Further, the logarithm N of the Dehn twist is $\underline{k}$-invariant,

PROP:

$$N = \sum_{m \geq 0} \overbrace{(2m-1) \frac{B_{2m}}{(2m)!}}^{\text{constant depends on normalizations}} e_{2m}$$

Define the Lie algebra of <u>special</u> <u>elliptic</u> <u>derivations</u> by

$$\mathrm{Der}^{s'} \mathbb{L}(A,T)^\wedge$$

$$:= \left\{ \delta \;\middle|\; \begin{array}{l} (1)\ \delta \text{ preserves the image of } \mathbb{L}(X_0, X_1)^\wedge \longrightarrow \mathbb{L}(A,T)^\wedge; \\ (2)\ \delta|_{\mathbb{L}(X_0,X_1)^\wedge} \text{ is special à la Ihara;} \\ (3)\ \delta \text{ commutes with } N. \end{array} \right\}$$

We have a homom

$$\underline{k} \hookrightarrow \mathrm{Der}^S \mathbb{L}(A,T)^\wedge.$$

This normalizes the image of $\underline{u}^{eis}$ (= Lie alg gen by $\{\varepsilon_{2n} : n \geq 1\}$ & $\mathfrak{sl}_2$.)

And an injection

$$\underline{k} \hookrightarrow \mathrm{Der}^S \mathbb{L}(A,T)^\wedge / \mathrm{im}\, \underline{u}^{eis} \cap \mathrm{Der}^S$$

<u>Questions:</u>

1. How big is $\mathrm{Der}^S \mathbb{L}(A,T)^\wedge$?

   Can one characterize its elements

2. How is it related to Enriquez's $\mathfrak{grt}_{\underline{1}}^{ell}$? ← elliptic GRT Lie alg

3. How big is $\underline{u}^{eis} \cap \mathrm{Der}^S \mathbb{L}(A,T)^\wedge$?

   Trivial ?

4. Is $\underline{k} \hookrightarrow \mathrm{Der}^S \mathbb{L}(A,T)^\wedge$ an isomorphism ?