THE HOMOLOGICAL CONJECTURES: PAST, PRESENT, AND FUTURE

MELVIN HOCHSTER

1. Disclaimer

This is the transcription of a lecture given by Melvin Hochster for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture introduces the homological conjectures, and in particular discusses several equivalent forms of the direct summand conjecture.

2. Direct Summand Conjecture

2.1. **Equivalent Forms.** We refer to [16, 17, 18, 19] for background.

Conjecture 2.1.1 (Direct Summand)**.** *If R is a regular ring and $R \subseteq S$ is a module-finite ring extension (i.e. the natural structure of an R-module on S makes S finitely generated as an R-module), then $R \subseteq S$ is a direct summand of S.*

By replacing S by S/P for a suitable minimal prime, we may assume that S is a domain as well.

If R is a Noetherian domain, we say that R is a *splinter* if $R \hookrightarrow S$ splits for all S that are module-finite over R. We now know that all regular Noetherian rings are splinters: this is the content of the direct summand conjecture, which is now proved [1, 2, 7]. Note that the characteristic $p > 0$ case was proved in [16] and the dimension 3 case in [14].

Observe that for Noetherian domains R containing the rational numbers, R is a splinter if and only if R is normal. One may use a multiple of the field trace map on their fraction fields to retract S to R when S is a module-finite domain extension of the normal domain R.

In studying the direct summand conjecture:

Remark 2.1.2. (1) One can assume R is local, by showing that R is a direct summand of S if and only if for each maximal ideal $\mathfrak{m} \subseteq R$, $R_\mathfrak{m} \to S_\mathfrak{m}$ makes $R_\mathfrak{m}$ a direct summand of $S_\mathfrak{m}$.

Date: March 12, 2018.

[1] Typeset by Ashwin Iyengar (aiyengar@berkeley.edu).

(2) Now assume R is local. One can assume R is complete as well, by showing that R is a direct summand of S if and only if $\widehat{R} \to S \otimes_R \widehat{R}$ makes $\widehat{R}$ into a direct summand of $S \otimes_R \widehat{R}$, where
$$\widehat{R} = \varprojlim_n R/\mathfrak{m}_R^n$$
is the completion of R along its maximal ideal.

Using the assumptions in 2.1.2, assume that R is a complete regular local ring. Then we may state the direct summand conjecture in the following equivalent way:

Conjecture 2.1.3 (Direct Summand, 2nd version). *Suppose R is a complete regular local ring. Let S be a module-finite extension of R. Then for any regular system of parameters $x_1, \ldots, x_n$ in R (i.e. $\dim R = n$ and $\mathfrak{m} = (x_1, \ldots, x_n)$) we have*
$$x_1^t \cdots x_n^t \notin (x_1^{t+1}, \ldots, x_n^{t+1})S$$
for all $t > 0$.

We will now give an indication as to the equivalence of these two statements. Let E denote the local cohomology module $H^n_{(x_1,\ldots,x_n)}(R)$, which is also the injective hull of the residue class field K of R. E is the directed union of the cyclic modules R/I_{t+1} where $I_{t+1} = (x_1^{t+1}, \ldots, x_n^{t+1})R$. Note also that $x_1^t \cdots x_n^t$ generates the one-dimensional (as a vector space) socle of I_{t+1}. Then:

$$\begin{aligned} E \to E \otimes_R S \text{ injective} &\iff \mathrm{Hom}_R(S \otimes_R E, E) \to \mathrm{Hom}_R(E, E) \text{ surjective} \\ &\iff \mathrm{Hom}_R(S, \mathrm{Hom}_R(E, E)) \to \mathrm{Hom}(E, E) \text{ surjective} \\ &\iff \mathrm{Hom}_R(S, R) \to R \text{ surjective} \\ &\iff R \hookrightarrow S \text{ splits.} \end{aligned}$$

Since E is the directed union of the R/I_{t+1}, the map $E \to E \otimes_R S$ is injective if and only if all the maps $R/I_{t+1} \to R/I_{t+1} \otimes_R S \cong S/I_{t+1}S$ are injective, and so for the splitting we need precisely that for all of these maps, the socle generator does not map to 0. This is equivalent to requiring that $x_1^t \cdots x_n^t$ is not in $(x_1^{t+1}, \ldots, x_n^{t+1})S$ for all t.

It turns out that the direct summand conjecture is equivalent as well to the following conjecture, known as the “monomial conjecture."

Conjecture 2.1.4 (Direct Summand, 3rd version). *Let $x_1, \ldots, x_n$ be a system of parameters for an arbitrary Noetherian local ring S of dimension n. Then we have*
$$x_1^t \cdots x_n^t \notin (x_1^{t+1}, \ldots, x_n^{t+1})S$$
for all $t > 0$.

It turns out not to matter whether or not the parameters are also parameters in a regular local subring R of S such that S is module-finite over R.

A fourth equivalent form of the conjecture goes as follows

Conjecture 2.1.5 (Direct Summand, 4th version)**.** *In a Noetherian ring R, the equation $x_1^t \cdots x_n^t - \sum_{i=1}^n y_i x_i^{t+1}$ has no solution such that $(x_1, \dots, x_n)R$ has height n as an ideal.*

Some intuition for this fourth formulation comes from the case where R is the following k-algebra

$$T = \frac{k[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]}{(x_1^t \dots x_n^t - \sum_{i=1}^n y_i x_i^{t+1})}.$$

Then R is an integral domain for $n > 1$ and a UFD for $n > 2$. Here, the ideal $(x_1, \dots, x_n)$ has height $n-1$. One may think of height geometrically as codimension. The issue is whether the codimension can increase to n in a homomorphic image of the hypersurface T, and one may think of this as a question about the behavior of the intersection of the hypersurface corresponding to T with a variety in the ambient space. In mixed characteristic one may replace k by a complete discrete rank one valuation domain.

Many other statements have been shown to be equivalent to the direct summand conjecture. We record some of these here without comment. If R is a domain we denote by R^+ the integral closure of R in an algebraic closure of its field of fractions, which is unique up to non-unique R-isomorphism. (This is different from the use of $^+$ in some later talks.)

Conjecture 2.1.6. *If (R, m) is a complete local domain of Krull dimension d, then $H_m^d(R^+) \neq 0$.*

See [18].

Conjecture 2.1.7 (Canonical element conjecture)**.** *Let (R, m, K) be a complete local ring. Extend the identity map on $R = K_1 = G_1$ to a map from the Koszul complex $K_\bullet$ of a system of parameters $x_1, \dots, x_d$ to a minimal free resolution $G_\bullet$ of K. The induced map $\phi : R \cong K_d$ into G_d has values in Ker$(G_d \to G_{d-1})$, which is a d th module of syzygies M of K. Then $\phi(1) \notin (x_1, \dots, x_d)S$.*

See [9, 18].

Conjecture 2.1.8 (Strong intersection conjecture)**.** *Let (R, m, K) be a local ring, let $0 \to G_n \to \cdots \to G_0$ be a finite complex of finitely generated free R-modules. Suppose that $H_0(G_\bullet) \neq 0$ has a minimal generator killed by a power of m, and that $H_i(G_\bullet)$ has finite length for $i \geq 1$. Then $\dim R \leq n$.*

This conjecture has also been referred to as the "improved new intersection conjecture."

This was proved by Evans-Griffith [10] if R has a big Cohen-Macaulay module. They used the strong intersection conjecture to show the following in equal characteristic:

Conjecture 2.1.9 (Evans-Griffith syzygy conjecture)**.** *A k th module of syzygies of a finitely generated module over a regular local ring, if not free, has torsion-free rank at least k.*

In [18] it is shown that the direct summand conjecture is equivalent to the canonical element conjecture and implies the strong intersection conjecture (and, therefore, the Evans-Griffith syzygy conjecture). Sankar Dutta [9] showed that the strong intersection conjecture is equivalent to the canonical element conjecture and, hence, to the direct summand conjecture.

Conjecture 2.1.10 (Descent of flatness for integral extensions)**.** *Let $R \to S$ be an integral extension of a Noetherian ring R and let M be any R-module. If $S \otimes_R M$ is S-flat, then M is R-flat.*

This was conjectured by Raynaud-Gruson and shown equivalent to the direct summand conjecture by Ohi.

Of course, all of these conjectures are now theorems.

3. Peskine-Szpiro Conjectures

We now list seven conjectures studies by Peskine and Szpiro [23]. First, we fix some conventions.

Let $(R, \mathfrak{m}_R, k)$ be a local Noetherian ring, $M \neq 0$ a finitely generated R-module with $\operatorname{pd} M < \infty$ (where pd denotes projective dimension over R), and $N \neq 0$ another finitely generated R-module (with no assumptions on $\operatorname{pd} N$). Given a nonzero finitely generated module Q over R, we use $\dim Q$ to denote the Krull dimension of $R/\operatorname{Ann} Q$. The following statements are all conjectures considered by Peskine and Szpiro. We note at once that (a), (e) and (f) all remain open at the time of this writing. The use of lettering agrees with that in the paper of Peskine and Szpiro, except for item (g). The grade of $M \neq 0$ is the length of a maximal regular sequence on R in $\operatorname{Ann} M$.

(a) $\dim M + \dim N \leq \dim R$ or $\dim N \leq \dim R - \dim M$.

(b) (M. Auslander) If $x_1, \ldots, x_n \in \mathfrak{m}_R$ is a regular sequence on M, then it is a regular sequence on R.

(c) (M. Auslander's rigidity conjecture) If $\operatorname{Tor}_i(M, Q) = 0$ then $\operatorname{Tor}_j(M, Q) = 0$ for all $j \geq i$.

(d) (Intersection conjecture (now a theorem)) $\dim N \leq \operatorname{pd} M$.

(e) $\dim N \leq \operatorname{grade} M$.

(f) $\dim M + \operatorname{grade}(M) = \dim R$.

(g) (Bass's conjecture.) If N has finite injective dimension, then R is Cohen-Macaulay.

Note that (d) may viewed as generalization of Krull's principal ideal theorem, perhaps the first really deep theorem in the theory of Noetherian rings:

Theorem 3.0.1 (Krull)**.** *If $\mathfrak{p}$ is a minimal prime of a principal ideal $(f) \subseteq R$ in a Noetherian ring R, then the height of $\mathfrak{p}$ is ≤ 1.*

This immediately reduces to the case where R is local: the height of $\mathfrak{p}$ is then $\dim R_{\mathfrak{p}}$. We may also reduce at once to the case where R is a domain by killing a suitable minimal prime and to the case where $f \neq 0$. Then f is not a zerodivisor, and $\operatorname{pd} R/fR = 1$. Taking $M = R/fR$ and $N = R$, we see that in this case the statement in (d) is that $\dim R \leq 1$, which means that (d) becomes Krull's principal ideal theorem.

From a geometric perspective, Krull's theoem says that every component of the intersection of a hypersurface with a variety of dimension d has dimension $\geq d-1$, which is an intersection theorem. This may help to explain why Peskine and Szpiro refer to (d) as "the intersection conjecture."

M. Auslander conjectured (c) and (b) and proved that $(c) \implies (b)$. His argument is intricate. Ray Heitmann proved that (c) is false in general in [13]. Note, however, that (c) is an open question if both M and N are assumed to have finite projective dimension.

Peskine-Szpiro proved that $(e) \iff (a)+(f)$, and that $(e) \implies (d)$, and $(d) \iff (b)$. They also showed that $(d) \implies (g)$ (Bass's conjecture [5]). Furthermore, they actually proved (d) in characteristic p and many equal characteristic 0 cases by reduction to characteristic p. This proved Bass's conjecture in the same cases. Later, in studying big Cohen-Macaulay modules [17] I observed that by using Artin approximation on the ring structure, one can deduce (d) and (b) in general in equal characteristic 0 from the case of prime characteristic $p > 0$.

Paul Roberts, using the Fulton-MacPherson-Baum theory of localized Chern classes [11] and, independently, H. Gillet and C. Soulé in [12], using related ideas in K-theory, prove (d) (and, hence (b)) in the general case. Paul Roberts also proved the intersection theorem in a strengthened form (obtained by Peskine and Szpiro in [24]) in equal characteristic 0 (without needing characteristic $p > 0$ methods) the Grauert-Reimenschneider vanishing theorem. See [26, 27, 28, 29, 30].

4. Big Cohen-Macaulay Modules and Algebras

First, an exercise:

Exercise 4.0.1. *If $x_1, \ldots, x_n$ is a regular sequence over an R-module M for some ring R, then*

(1) $x_1^t \cdots x_n^t \in (x_1^{t+1}, \ldots, x_n^{t+1})$, and

(2) If $u \in M \setminus (x_1, \ldots, x_n)M$, then

$$x_1^t \cdots x_n^t u \notin (x_1^{t+1}, \ldots, x_n^{t+1})M$$

Here, M need not be finitely generated.

This shows that a module-finite local extension $R \to S$ with R regular splits if the image of a regular system of parameters in R is a regular sequence on some S-module M.

Thus, one wants to show that every local ring S has a "big Cohen-Macaulay module," i.e., a module M such that a system of parameters for S is a regular sequence on M. The existence of big Cohen-Macaulay modules for all complete local domains implies both the direct summand conjecture and the intersection conjecture. Note that if M is such that one system of parameters for R is a regular sequence, then every system of parameters of R is a regular sequence on the m-adic completion of M: see [4]. When this condition is satisfied, M is called a *balanced* big Cohen-Macaulay module.

I showed big Cohen-Macaulay modules exist in characteristic p in the early 1970s, as well as in equal characteristic 0, by reduction to characteristic p, utilizing the idea of applying Artin approximation to the ring structure as well as to solving certain systems of equations. See [19].

If $R \to S$ is a local map of complete local integral domains, one would like to know that there exists B, a big Cohen-Macaulay algebra over R, and C, a big Cohen-Macaulay algebra over S, with a map $B \to C$ such that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

commutes. This is referred to as the existence of "weakly functorial" big Cohen-Macaulay algebras.

In fact, if R is a complete local integral domain in characteristic $p > 0$, then define R^+ to be the integral closure of R in the algebraic closure of the fraction field of R, which is unique up to non-unique isomorphism. (This notation differs from the use of $^+$ in some later talks in this workshop.) It was shown by Craig Huneke and myself that this is a big Cohen-Macaulay algebra, and that $R \mapsto R^+$ is weakly functorial in the sense that there is a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & S \arrow[d, hook] \\ R^+ \arrow[r] & S^+ \end{tikzcd}$$

and from this it follows as well that weakly functorial big Cohen-Macaulay algebras exist for maps of complete local domains containing the rationals as well, by reduction to prime characteristic $p > 0$. See

In recent work, Ray Heitman and Linquan Ma [15] proved many cases of the existence of weakly functorial Cohen-Macaulay algebras, and Yves André [3] has recently settled the general case.

5. The vanishing conjecture for maps of Tor

The following result was proved by Craig Huneke and myself in equal characteristic (first using tight closure and reduction to characteristic $p > 0$, and later using the Cohen-Macaulay property of R^+ for the characteristic p case in [21]).

André's recent result [3] yields this conjecture in general.

Conjecture 5.0.1 (vanishing conjecture for maps of Tor)**.** *Let $A \to R \to S$ be maps of Noetherian rings, where A and S are regular and R is module-finite and torsion-free over A. Then for every A-module M, the map $\mathrm{Tor}_i^A(M,\, R) \to \mathrm{Tor}_i^A(M,\, S)$ is 0 for all $i \geq 1$.*

This implies both the direct summand conjecture and the statement that direct summands of regular rings are Cohen-Macaulay, a generalization of the result of [22]. The fact that direct summands of regular rings are Cohen-Macaulay already follows from the recent work of Heitman and Ma [15], which depends on the equivalence of the vanishing conjecture for maps of Tor with the strong direct summand conjecture of Ranganathan (the equivalence is proved in [25]), as well as from the more recent work of André) [3], and therefore is in a position of great importance among the homological conjectures. All of these conjectures are now theorems.

6. Serre multiplicities

Serre [31] defined an intersection multiplicity for finitely generated modules $M, N \neq 0$ over a regular local ring (R, m, K) such that $\ell(M \otimes_R N)$ is finite (i.e., the intersection of the supports of M and N is $\{m\}$) as follows:

$$\chi(M,\, N) := \sum_{i=0}^{\dim R} (-1)\ell\big(\mathrm{Tor}_i^R(M,\, N)\big).$$

Serre proved that if $\widehat{R}$ is formal power series over a discrete valuation ring or field (this will hold if R is equicharacteristic regular or mixed characteristic p regular and $p \notin m^2$) then one has

(a) $\dim M + \dim N \leq \dim R$.

(b) If $\dim M + \dim N < \dim R$ then $\chi(M, N) = 0$.

(c) If $\dim M + \dim N = \dim R$ then $\chi(M, N) > 0$.

Serre also proved that (a) holds whenever R is regular local. Serre asserted that it is natural to conjecture that (b) and (c) are true for every regular local ring.

Paul Roberts and, independently, Gillet and Soulé, proved that (b) holds for all regular local rings. Gabber [6] proved that $\chi(M, N) \geq 0$ for all regular rings using results of de Jong [8] on alterations. But (c) remains an open question for ramified regular local rings in dimension ≥ 5.

7. Open questions

As already mentioned, part (c) in the preceding section is open in general for ramified regular local rings, i.e. mixed characteristic p regular local rings such that p is in the square of the maximal ideal. It would follow if one had the existence of faithful finitely generated Cohen-Macaulay modules over complete local domains, but this remains open in dimension 3 and higher in all characteristics. The conjectures (a), (e), and (f) considered by Peskine and Szpiro remain open in all characteristics. The characterization of Noetherian domains that are splinters is an open question both in characteristic $p > 0$ and in mixed characteristic. In characteristic p, it may be that splinters coincide with weakly F-regular rings (rings in which every ideal is tightly closed): this is true if the ring is Gorenstein and in several other special cases. A great deal remains to be done.

References

[1] Y. André, *La lemme d'Abhyankar perfectoide*, preprint, 55 pp. arXiv:1609.00320 [math.AG] August 31, 2016.
[2] Y. André, *La conjecture du facteur direct*, preprint, 15 pp. arXiv:1609.00345 [math.AG] September 1, 2016.
[3] Y. André, *Weak functoriality of Cohen-Macaulay algebras*, preprint, 12pp. arXiv:1801.10010 [math.AC] January 30, 2018.
[4] J. Bartijn and J. R. Strooker, *Modifications monomiales*, Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982), 192–217, Lecture Notes in Math. **1029**, Springer, Berlin, 1983.
[5] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963) 8–28.
[6] P. Berthelot, *Altérations de variétes algébriques [d'après A. J. de Jong]*, Séminaire Bourbaki, 48ème année, n^{O} 815, pp. 815–01 – 815–39.
[7] B. Bhatt, *On the direct summand conjecture and its derived variant*, preprint, 2018, arXiv: 1608.2v1 [math.AG].
[8] A. J. de Jong, *Smoothness, semistability, and alterations*, Publ. Math. I.H.E.S. **83** (1996) 51–93.
[9] S. P. Dutta, *On the canonical element conjecture*, Trans. Amer. Math. Soc. **299** (1987) 803–811.
[10] G. Evans and P. Griffith, *The syzygy problem*, Annals of Math. **114** (1981) 323–333.
[11] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984.
[12] H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Acad. Sci. Paris Série I no. 3 t. **300** (1985) 71–74.
[13] R. C. Heitmann, *A counterexample to the rigidity conjecture for rings*, Bull. Amer. Math. Soc. New Series **29** (1993) 94–97.
[14] R. C. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. (2) **156** (2002) 695–712.
[15] R. C. Heitmann and L. Ma, *Big Cohen-Macaulay Algebras and the vanishing conjecture for maps of Tor in mixed characteristic*, preprint, 2017, arXiv:1703.08281v1 [math.AC].
[16] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43.
[17] M. Hochster, *Topics in the homological theory of modules over commutative rings*, C.B.M.S. Regional Conf. Ser. in Math. No. **24**, Amer. Math. Soc., Providence, R.I., 1975.
[18] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra **84** (1983) 503–553.
[19] M. Hochster, *Homological conjectures and lim Cohen-Macaulay sequences*, in Proceedings of the Conference on Homological and Computational Methods in Commutative Algebra (honoring Winfried Bruns, Cortona Italy, 2016), 2018.

[20] M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992) 53–89.

[21] M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*, Advances in Math. **113** (1995) 45–117.

[22] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math., **13** (1974) 115–175.

[23] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I.H.E.S. (Paris) **42** (1973) 323–395.

[24] C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A **278** (1974) 1421–1424.

[25] N. Ranganathan, *Splitting in module-finite extension rings and the vanishing conjecture for maps of Tor*, Thesis, University of Michigan, 2000.

[26] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. Éc. Norm. Sup. **9** (1976) 103–106.

[27] P. Roberts, *Cohen-Macaulay complexes and an analytic proof of the new intersection conjecture*, J. Algebra **66** (1980) 225–230.

[28] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. **13** (1985) 127–130.

[29] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sc. Paris Sér. I **304** (1987) 177–180.

[30] P. Roberts, *Intersection theorems*, in Commutative Algebra, Math. Sci. Research Inst. Publ. **15** Springer-Verlag, New York-Berlin-Heidelberg,1989, 417–436.

[31] J.-P. Serre, Algèbre locale · Multiplicités, Springer-Verlag Lecture Notes in Math. No. 11, Séconde Edition, 1965.