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Program: Perfectoid fields: definition, Fontaine's tilting functor $R \mapsto R^\flat$ for general rings, tilt $K^\flat$ of a perfectoid field K and relating continuous valuations on K and $K^\flat$, statement of Fontaine-Wittenberger theorem and the proof in the algebraically closed case. *References:* [Sc1, Section 3], [Bh3, Section 3], [Co, parts of Lecture 17].

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Request: the reader is kindly requested to inform me if there are typos and/or inaccuracies, so I can correct.

1. Perfectoid fields

1.1. **Valued field.** Recall a *valuation* $|\ |: K \to \Gamma \cup \{0\}$ on a field K in a totally ordered multiplicative group Γ, where we set $0 < c\ \forall c \in \Gamma$, is a group homomorphism on $K^\times$ with the NA (non-archimedean) property $|x+y| \le \max(|x|, |y|)$. It has rank 1 if the *value group* $|K^\times|$ is isomorphic as an ordered group to a subgroup of $(\mathbb{R}_{>0}, \times)$ and is discrete if $|K^\times|$ is isomorphic as an ordered group to $\mathbb{Z}$. Two valuations are equivalent if there is an ordered isomorphism of the value groups which transforms one valuation into the other. Then isomorphism classes of valuations are in one to one correspondence with *valuation subrings* $R \subset K$, where a subring is a valuation subring if for all $x \in K^\times$ either $x \in R$ or $x^{-1} \in R$. To $|\ |$ one assigns $R = \{x, |x| \le 1\}$ and to R one assigns the residue homomorphism $K^\times \to K^\times/R^\times$ where the order on $K^\times/R^\times$ is defined by $\bar{x} \le \bar{y}$ iff $xy^{-1} \in R$ for any lifts $x, y \in K^\times$ of $\bar{x}, \bar{y}$.

1.1.1. *Notations.* Let K be a non-archimedean field. One denotes by $K^0 = \{x, |x| \le 1\}$ its *valuation ring*, by $K^{00} = \{x, |x| < 1\}$ its maximal ideal and by $k = K^0/K^{00}$ its *residue field*. Then $K^{0\times} = K^0 \setminus K^{00}$.

1.1.2. $|\ |$ is *non-trivial* if $\exists a \in K^\times$ with $|a| \neq 1$. This is equivalent to $K^0 \subset K$ being strict.

1.1.3. A valuation $|\ |$ on K defines a topology with basis of neighbourhoods of 0 being the balls $\{x \in K, |x| < a\}$, and $(K, |\ |)$ is *complete* if the topological space is complete, i.e. if Cauchy sequences converge.

1.2. **Perfectoid fields.**

1.3. **Definition.** A *perfectoid field* K is a field which is endowed with a non-trivial, non-discrete rank 1 NA valuation, which is complete, and such that the Frobenius map

$$\Phi : K^0/p \to K^0/p,\ x \mapsto x^p$$

is *surjective*.

1.4. **Valued group.** As $1 \in K^0$, so $p \in K^0$ as it is a subring. As p dies in K^0/p, $p \notin K^{0\times}$ thus $|p| < 1$ i.e. $p \in K^{00}$.

Lemma 1. *$|K^\times|$ spanned as a multiplicative group by $|p|^{\mathbb{Z}}$ and $\{|x|\}$ for $|p| < |x| \leq 1$. Furthermore, there are elements $x \in K^0$ such that $|p| < |x| < 1$. For any x such that $|x| < 1$ one has $K = K^0[x^{-1}]$.*

Proof. $|K^\times|$ spanned by $|K^{00}|$. If $0 \neq x \in K^{00}$ with $|x| \leq |p|$ then $\exists n \in \mathbb{N}_{>0}$ with $|p|^n \leq |x| < |p|^{n-1}$ thus $p^n = xy$ with $|p| < |y| \leq 1$ and $|y| = 1$ iff $|x| = |p|^n$. This proves the first part. Further, if $|x| = |p^n|$, $n \in \mathbb{N}$ for all $x \in K^0$, then $|K^\times|$ is discrete, which is excluded by definition. This proves the second part. Finally, if $|z| > 1$ then as $|x^{-1}| > 1$ there is a $n \in \mathbb{N}$ such that $|x^{-n}| > z$ which is to say $zx^n \in K^0$. This finishes the proof.

□

Lemma 2. *K perfectoid field then $|K^\times|$ as a multiplicative group is p-divisible.*

Proof. 1) First we prove that for $|p| < |x| \leq 1$, on has $|x| = |y|^p$ for some $y \in K^0, |p| < |y| \leq 1$. Indeed, $\bar{x} = \bar{y}^p$, for some $y \in K^0$, y^p not divisible by p, so $|p| < |y|^p$. Equation $\bar{x} = \bar{y}^p$ is equivalent to $x = y^p + pz$ for some $z \in K^0$. Thus NA implies $|x| \leq \max(|y|^p, |p||z|) = |y|^p$, and $|y|^p \leq \max(|x|, |p||z|) = |x|$. Thus $|x| = |y|^p$. Then by Claim 1 enough to prove $|p|^{\mathbb{Z}}$ is p-divisible. Take $x \in K^0$ with $|p| < |x| < 1$, then $p = xy$ for some $y \in K^{00}$ with $|p| < |p/x| = |y| < 1$. Thus both $|x|$ and $|y|$ are p-powers.

□

Remark 3 (*Not needed in the lecture*)**.** K perfectoid field then

1) $(K^{00})^2 \xrightarrow{\subset =} K^{00}$.
2) K^0 not noetherian.

Proof. 1) From 1), for any $x \in K^{00}$ there is $y \in K^{00}$ such that $|x| = |y^p|$, i.e. $x = y^p u$, $u \in K^{0\times}$. As $p \geq 2$ this shows 1). 2) If K^0 was noetherian, then $K^{00} = (K^{00})^2$ would imply $K^{00} = 0$ by Nakayama, thus $K^0 = (K^0)^\times$ thus $K^0 = K$ and the valuation would be trivial, a contradiction. □

1.5. **If** $\mathrm{char}(K) = p > 0$**,** then $K^0/p = K^0$ and K^0 is perfect iff K is perfect. Thus a perfectoid char. $p > 0$ field is a *complete perfect non-archimedean field*.

2. Tilting functor

2.1. **Tilt of a ring.**

Definition 4. A char. $p > 0$ ring R is *perfect* if the Frobenius (which is an endomorphism as the char. is $p > 0$)

$$\Phi : R \to R, \ x \mapsto x^p$$

is an isomorphism. It is *semiperfect* if it is surjective.

Definition 5. If R is *any* ring, Fontaine's tilt is defined to be

$$R^\flat := (R/p)^{\mathrm{perf}} := \varprojlim_{\Phi}(R/p)$$

as a *topological* ring with R/p being discrete and the topology on $R^\flat$ being the inverse limit topology for which a basis of topology consists of $\{(x_n)_n, n \in \mathbb{N}, \Phi(x_{n+1}) = x_n, x_{n_0} \in \Sigma\}$ where $\Sigma \subset R/p$ is a given subset.

2.2. **Tilt of a ring is perfect.**

Lemma 6. *Assume char. $R = p > 0$. Then R^{perf} is perfect, and if $S \to R$ is an homomorphism of rings with S perfect of char. $p > 0$, then it factors through $S \to R^{\mathrm{perf}} \to R$.*

Proof. Has $(x, x^{1/p}, x^{1/p^2}, \ldots) = (x^{1/p}, x^{1/p^2}, \ldots)^p$ and $(x, x^{1/p}, x^{1/p^2}, \ldots)^p = 0$ iff $x^p = x = x^{1/p} = \ldots = 0$ iff $(x, x^{1/p}, \ldots) = 0$. If S has char. $p > 0$ then if $x \in S$ with $x = y^p, \exists y$ then y is unique as $x^p = y^p$ iff $(x-y)^p = 0$ and S perfect implies $x = y$. So any $f : S \to R$ with S perfect of char. $p > 0$ lifts uniquely to $S \to R^{\mathrm{perf}} \to R$.

□

2.3. **Fontaine construction.** Let K be a perfectoid field. Let $\pi \in K^\times$ with $|p| \le |\pi| < 1$ (see above for existence). Then $p = \pi\nu$ for some $\nu \in K^0$ thus $K^0/p \twoheadrightarrow K^0/\pi$ thus K^0/π is a semiperfect char. $p > 0$ ring. We define

$$(K^0/\pi)^{\mathrm{perf}} (:= \varprojlim_{\Phi} K^0/\pi)$$

as a topological ring which is perfect. One has a projection map

$$\mathrm{proj} : \varprojlim_{x \mapsto x^p} K^0 \to (K^0/\pi)^{\mathrm{perf}}.$$

If char. K^0 is $p > 0$, this is an homomorphism, if char. K^0 is 0, it is a multiplicative map.

Lemma 7. *1) The projection proj is an homeomorphism. In particular, $(K^0/\pi)^{\mathrm{perf}}$ as a multiplicative topological set is independent of the choice of π. Moreover, there is a continuous multiplicative map*

$$(K^0/\pi)^{\mathrm{perf}} \to K^0, \; x \mapsto x^\sharp.$$

2) There is an element $\pi^\flat \in (K^0/\pi)^{\mathrm{perf}}$ such that $|(\pi^\flat)^\sharp| = |\pi|$.

Proof. We prove 1). • Let $(\bar{x}_0, \bar{x}_1, \ldots) \in (K^0/\pi)^{\mathrm{perf}}$. Lift $\bar{x}_n$ to $x_n \in K^0$. Then

$$x^\sharp = (\text{Cauchy}) \lim_{n\to\infty} x_n^{p^n}$$

is well defined. Enough to see

Claim 8.

$$\mathrm{Im}\ x_n^{p^n} \in K^0/\pi^{n+1}$$

is well defined.

Proof. Write $x_n' = x_n + \pi z$. Thus

$$(x_n')^{p^n} = x_n^{p^n} + p^n \pi x_n^{p^n - 1} z + \binom{p^n}{2} \pi^2 x_{n-2}^{p^n-2} z^2 + \dots$$

so enough to see

$$|\binom{p^n}{i}| \le |p|^{n+1-i}$$

for $1 \le i \le n$. Either we know that $(1+px)^{p^n} \in$ 1-units in $\mathbb{Z}/p^{n+1}$ or else (Scholze) one computes the inequality for the valuations for $n = 0$ where it is trivial, then inductively $(x_n')^{p^i} - x_n^{p^i} = (x_n^{p^{i-1}} + ((x_n')^{p^{i-1}} - x_n^{p^{i-1}})))^p - x_n^{p^i}$ for $0 < i \le n$. $\square$

• The map $(K^0/\pi)^{\mathrm{perf}} \to K^0,\ x \mapsto x^\sharp$ is continuous by definition.
• The inverse to proj is defined by

$$\mathrm{proj}^{-1} : (K^0/\pi)^{\mathrm{perf}} \to \varprojlim_{x \mapsto x^p} K^0,$$
$$x = (\bar{x}_0, \bar{x}_1, \dots) \mapsto ((\bar{x}_0, \bar{x}_1, \dots)^\sharp, (\bar{x}_1, \bar{x}_2, \dots)^\sharp \dots) = (x^\sharp, (x^{1/p})^\sharp, \dots))$$

It is continuous as $x \mapsto x^\sharp$ is continous.
• We prove 2). As the valuation group is p-divisible, there is a $\pi_1 \in K^0$ such that $|\pi_1|^p = |\pi| < 1$, thus $1 > |\pi_1| > |\pi|$ one has $\pi = \pi_1 \gamma$ for some $\gamma \in K^{00}$ thus $0 \ne \bar{\pi}_1 \in K^0/\pi$. As $K^0/p \twoheadrightarrow K^0/\pi$, and the Frobenius is surjective on K^0/p, it is surjective on K^0/π thus $\bar{\pi}_1$ lifts to

$$\pi^\flat := (0, \bar{\pi}_1, \dots) \in (K^0/\pi)^{\mathrm{perf}},$$

of course non-uniquely. Has from Claim 8 $(\pi^\flat)^\sharp = \pi_1^p + \pi^2\gamma$ for some $\gamma \in K^0$, thus $|(\pi^\flat)^\sharp| \le \max(|\pi|, |\pi^2\gamma|) = |\pi|$, and $|\pi^2\gamma| < |\pi| \le \max(|(\pi^\flat)^\sharp|, |\pi^2\gamma|)$, thus $\max(|(\pi^\flat)^\sharp|, |\pi^2\gamma|) = |(\pi^\flat)^\sharp|$. Thus $|\pi| = |(\pi^\flat)^\sharp|$. This finishes the proof. $\square$

3. Tilt of a perfectoid field

3.1. **Definition of $K^\flat$.** The *tilt* of the perfectoid field K is

$$K^\flat = (K^0/\pi)^{\mathrm{perf}}[(\pi^\flat)^{-1}].$$

Lemma 9. *One has an extension*

$$\begin{tikzcd}
(K^0/\pi)^{\mathrm{perf}} \arrow[r, "(\mathrm{proj})^{-1}"] \arrow[d] \arrow[rr, bend left, "\sharp"] & \varprojlim_{x \mapsto x^p} K^0 \arrow[r, "(x_0, x_1, \dots) \mapsto x_0"'] \arrow[d] & K^0 \arrow[d] \\
K^\flat \arrow[r, "(\mathrm{proj})^{-1}"] \arrow[rr, bend right, "\sharp"'] & \varprojlim_{x \mapsto x^p} K \arrow[r, "(x_0, x_1, \dots) \mapsto x_0"] & K
\end{tikzcd}$$

Then $K^\flat$ is a perfectoid field of char. $p > 0$.

$$K^{\flat 0} = \varprojlim_{x \mapsto x^p} K^0 \cong \varprojlim_{\Phi} K^0/\pi.$$

The rank 1 valuation on $K^\flat$ is defined by

$$|x|_{K^\flat} = |x^\sharp|_K$$

and on the valuation groups one has

$$|K^{\flat\times}| = |K^\times|.$$

Moreover

$$K^{0\flat}/\pi^\flat \cong K^0/\pi$$

and on the residue fields one has

$$K^{\flat 0}/K^{\flat 00} = K^0/K^{00}.$$

Finally if K is of char. $p > 0$, one has $K^\flat = K$.

Proof. • Since proj and the projection to the first coordinate are multiplicative, they extend to $K^\flat$. Recall that one has $K = K^0[1/\pi]$. The image of proj^{-1} on $K^\flat$ is then $\varprojlim_{x \mapsto x^p} K$ thus the map is bijective. Now it is an homeomorphism on $(K^0/\pi)^{\text{perf}}$. Since the topology is compatible with the ring structure, thus the field structure, it is an homeomorphism on $K^\flat$.
• This already implies that $K^\flat$ is a field.
• Topology on $K^\flat$: the multiplicative map $K^\flat \to K, x \to x^\sharp$ is continuous as it is obtained from the map on $(K^0/\pi)^{\text{perf}}$ after inverting $\pi^\flat$. Moreover, this map determines the topology on $\varprojlim_{x \mapsto x^p} K$ and this defines the rank 1 valuation $K^\flat \to \mathbb{R}_{>0},\ x \mapsto |x^\sharp|$.
• $K^\flat$ is perfect via the multiplicative homeomorphism $K^\flat \to \varprojlim_{x \mapsto x^p} K$. It is complete as $(K^0/\pi)^{\text{perf}}$ is complete. It has char. $p > 0$ as $(K^0/\pi)^{\text{perf}}$ has char. $p > 0$.
• The valuation groups are by definition the same $|K^{\flat\times}| = |K^\times|$.
• $K^{\flat 0} = \{y, x = \text{proj}^{-1}(y), |x_0| \leq 1\}$ so comes from $(K^0/\pi)^{\text{perf}}$. That is

$$K^{\flat 0} = (K^0/\pi)^{\text{perf}}.$$

Then $\pi^\flat \mapsto 0 \in K^0/\pi$ thus $K^{\flat 0}/\pi^\flat \to K^0/\pi$, which is surjective as $K^0/p \to K^0/p,\ x \mapsto x^p$ is surjective, and $K^0/p \twoheadrightarrow K^0/\pi$ thus $K^0/\pi \to K^0/\pi,\ x \mapsto x^p$ is surjective. Thus $K^{\flat 0} = (K^0/\pi)^{\text{perf}} \to K^0/\pi$ is surjective. Let $x \in K^{\flat 0}$ mapping to $0 \in K^0/\pi$. Then by definition $|x| \leq |\pi| = |\pi^\flat|$ thus $x = \pi^\flat u, u \in (K^\flat)$. That is

$$K^{\flat 0}/\pi^\flat \to K^0/\pi$$

is an isomorphism.
• One has $K^{\flat 00} = \{y, x = \text{proj}^{-1}(y), |x_0| < 1\}$ thus

$$K^{\flat 0}/K^{\flat 00} = K^0/K^{00}.$$

• Finally if K has char. $p > 0$ all multiplicative homeomorphisms involved in the proof are isomorphisms. This finishes the proof.

$\square$

4. Continuous valuations of K and $K^\flat$

Proposition 10. *Let K be a perfectoid field with tilt $K^\flat$. Then for any continuous valuation $|\ | : K \to \Gamma$, the map*

$$K^\flat \to \Gamma,\ \ x \mapsto |x|^\flat := |x^\sharp|$$

is a continuous valuation on $K^\flat$. The assignment $|\ | \mapsto |\ |^\flat$ is a bijection on equivalences of continuous valuation on both sides.

Proof. • Writing $K^\flat = (K^0/\pi)^{\mathrm{perf}}[1/\pi]$ and for $x = (\bar{x}_0, \bar{x}_1, \ldots) \in (K^0/\pi)^{\mathrm{perf}}$ with $x^\sharp = \text{Cauchy} \lim_n x_n^{p^n}$ we see that $|x^\sharp| = 0$ iff $|x_n|^{p^n} \to 0$ iff $x_n^{p^n} \in (\pi)$ iff $x = 0$. Multiplicativity is clear. Similarly

$$\begin{aligned} |x+y|^\sharp &= \lim_{n\to\infty}[((x^{1/p^n})^\sharp + (y^{1/p^n})^\sharp]^{p^n} \\ \max\bigl(&\lim_{n\to\infty}|(x^{1/p^n})^\sharp|^{p^n}, \lim_{n\to\infty}|(y^{1/p^n})^\sharp|^{p^n}\bigr) \\ &= \max(|x^\sharp|, |y^\sharp|). \end{aligned}$$

• We have now to compare the set of continuous valuations modulo equivalence on both sides.
• A valuation $||\ || : K \to \Gamma$ is *continuous* iff whenever $f_n \to 0$ in K, i.e. $|f_n| \to 0$ in Γ, then $||f_n|| \to 0$ in Γ, iff

$$(K, |\ |) \to (K, ||\ ||) \text{ is continuous.}$$

In particular, for $f \in K^{00}$, $||f||^n \to 0$ thus $K^{00} \subset \mathfrak{m} \subset R$ where $\mathfrak{m} \subset R$ is the maximal ideal and the ring of the valuation $||\ ||$. In particular, the valuation $||\ ||$ can not be trivial. The condition $K^{00} \subset \mathfrak{m}$ is enough: assume $f_n \to 0$ thus there is a n_0 such that $f_n = f_{n_0}^{m(n)} u_n$ with $m(n) \to \infty$ as $n \to \infty$ and $u_n \in K^{00}$. This implies $||f_n|| = ||f_{n_0}||^{m(n)}||u_n|| \to 0$ as $n \to \infty$. So $||\ ||$ is continuous iff $K^{00} \subset \mathfrak{m}$.

Assume there is a non-zero element $x \in R \setminus K^0$. Then $|x| > 1$ thus $1/x \in K^{00}$ thus $1/x \in \mathfrak{m}$ thus $x^{-1} \in R^\times \cap \mathfrak{m}$. Thus this set is $= \emptyset$ as $||\ ||$ is not trivial. So finally $||\ ||$ is *continuous* iff

$$K^{00} \subset \mathfrak{m} \subset R \subset K^0.$$

[And from this one deduces that R is open, as it contains an open subgroup K^{00}.]
• So to a continuous valuation $||\ ||$, one associates the ring $R/K^{00} \subset K^0/K^{00}$. It is a valuation subring: let $x \in K^0 \setminus R \subset K^0 \setminus K^{00} = (K^0)^\times$. Then as R is a valuation ring, $x^{-1} \in R$. Vice-versa, if $\bar{\mathfrak{m}} \subset \bar{R} \subset K^0/K^{00}$ is a valuation subring, its inverse image verifies $K^{00} \subset \mathfrak{m} \subset R \subset K^0$. On the other hand, $R \subset K$ is a valuation subring: if $x \in K^0 \setminus R \subset (K^0)^\times$, then $\overline{x^{-1}} \in \bar{R}$ thus $x^{-1} \in R$. If $x \in K^\times \setminus K^0$ then $x^{-1} \in K^0$. By the previous either $x^{-1} \in R$

or $(x^{-1})^{-1} = x \in R$. Thus the valuation is continuous.
• To go to the tilt side, pass through $K^0/K^{00} = K^{\flat 0}/K^{\flat 00}$ and go up to $K^\flat$.
□

Remark 11. The *valuation group* of $|\ |$ is $K^\times/K^{0\times}$, the one of $|\ |_R$ is $K^\times/R^\times$, the one of $|\ |_{\bar{R}}$, for $\bar{R} = R/K^{00}$, is $(K^0/K^{00})^\times/\bar{R}^\times = (K^0)^\times/R^\times$.

5. Galois groups of K and $K^\flat$ are canonically the same

To this, we have to prove first

Proposition 12. *If $K^\flat$ is algebraically closed so is K.*

Proof. • It is enough to prove that an integral element over K^0 lies in K^0 for if $\pi \in K^{00}$ with $|\pi| < 1$ then if $X^d + \lambda_1 X^{d-1} + \ldots + \lambda_d,\ \lambda_i \in K$, then $\lambda_i = \pi^{n_i}\mu_i, \mu_i \in K^{00}$ so replacing X by $X\pi^m$ for some power m one obtains an irreducible polynomial

$$P(X) = X^d + a_1 X^{d-1} + \ldots + a_d,\ a_i \in K^0.$$

• $K \subset L$ Galois extension defined by the roots of P. Then the valuation extends to L and as K is complete, the extension is unique. We conclude that the Galois group of L/K fixes $|\ |$, thus all roots have the same valuation. Thus one has $|a_i| \leq |a_d|^{i/d}$ (the $\leq$ comes from possible cancellations by taking the symmetric functions of the roots). (If we were writing the valuation additively as v, this would be equivalent to saying that the Newton polygon, which is the convex full of the points $\{(0,0),(1,v(a_1)),\ldots,(d,v(a_d))\}$ is the line passing through 0 with slope 1.)
• As $K^\flat$ is algebraically closed, $|K^{\flat\times}|$ is divisible by d, so so is $|K^\times| = |K^{\flat\times}|$. Thus one has $a_d = \alpha^d u,\ u \in K^{0\times}$ thus $|a_d| = |\alpha|^d$. Thus

$$P(X) = \alpha^d[Y^d + a_1\alpha^{-1}Y^{d-1} + a_2\alpha^{-2}Y^{d-2}\ldots + u],\ Y = \alpha^{-1}X,\ a_i\alpha^{-i} \in K^0.$$

Thus may assume $|a_d| = 1$.
• Take

$$Q(X) = X^d + b_1 X^{d-1} + \ldots + b_d,\ b_i \in K^{\flat 0}$$

a lift of the residue class of P in $(K^0/\pi)[X] = (K^{\flat 0}/\pi^\sharp)[X]$. Let $y \in K^\flat$ be a root of Q. As $|y|^d \leq \max(|b_1||y|^{d-1},\ldots,|b_d|)$, has $|y| \leq 1$ so $y \in K^{\flat 0}$.
• Write

$$K^0[X] \ni P(X + y^\sharp) = X^d + c_1 X^{d-1} + \ldots c_d,\ c_d = P(y^\sharp).$$

Has $P(y^\sharp) = \pi\gamma_d,\ \gamma_d \in K^0$ and irreducibility again implies $|c_i| \leq (|\pi||\gamma_d|)^{i/d}$.
• Write again $\pi\gamma_d = c^d u,\ u \in K^{0\times}$, thus $|c_i c^{-i}| \leq 1$, then

$$c^{-d}P(cX + y^\sharp) = X^d + c_1 c^{-1} X^{d-1} + \ldots + u$$

lies in $K^0[X]$ and is again irreducible. Thus again looking at it mod π and taking a root y_1, then $y_1^\sharp$, one obtains $|c^{-d}P(cy_1^\sharp + y^\sharp)| \leq |\pi|$ that is $|P(cy_1^\sharp + y^\sharp)| \leq |\pi|^2$. Write $c = \gamma_1,\ |\gamma_1|^d = |\pi|$. So we found $y^\sharp$, then $\gamma_1 y_1^\sharp + y^\sharp$. Iterating like this, we find a sequence $\gamma_i \in K^0, |\gamma_i|^d \leq |\pi|^i$ and

$y_i^\sharp \in K^0$ with $|P(\gamma_n y_n^\sharp + \gamma_{n-1} y_{n-1}^\sharp + \ldots + \gamma_1 y^\sharp + y^\sharp)| \le |\pi|^{m(n)},\ m(n) \to \infty$ as $n \to \infty$. As $z_n = \gamma_n y_n^\sharp + \gamma_{n-1} y_{n-1}^\sharp + \ldots + \gamma_1 y^\sharp + y^\sharp$ is a Cauchy sequence, it converges to a root of P.

$\square$

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Hélène Esnault, Freie Universität Berlin