Almost Mathematics and the Homological Conjectures

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Outline

Almost Zero Modules

Definition. Let R be a ring (commutative, with 1) and let $\mathfrak{m}$ be an ideal of R such that $\mathfrak{m}^2 = \mathfrak{m}$. An R-module M is *almost zero* if $\mathfrak{m}M = 0$.

For Noetherian local rings R there are very few ideals $\mathfrak{m}$ with $\mathfrak{m}^2 = \mathfrak{m}$ (0 and R). The theory is interesting mostly for non-Noetherian rings.

A common example is the ideal generated by all g^{1/p^n}, where g is an element of R with a compatible set of p^nth roots for all integers n. This ideal will be denoted (g^{1/p^∞}). In most cases, g will be a non-zero-divisor.

Almost Mathematics

This definition, together with the term "Almost Mathematics", was introduced by Faltings in *Almost Étale Extensions* (1998). In this paper, the ideal $\mathfrak{m}$ was union of principal ideals $\mathfrak{m}_\alpha$ generated by elements π^α with $\pi^\alpha \pi^\beta = (unit)\pi^{\alpha+\beta}$.

He also proved the first version of the Almost Purity Theorem. This theorem implies that for certain extensions in mixed characteristic p, $R \subseteq S$, if $S[1/p]$ is étale over $R[1/p]$, then S is almost étale over R. In particular, S is almost flat over R.

Heitmann's Theorem

In 2002 Heitmann proved the Direct Summand Conjecture in dimension 3.

The basic situation is that R is a complete regular local ring of mixed characteristic p and S is a finite extension of R which is a normal domain. We let S^+ be the integral closure of S in the algebraic closure of its quotient field.

Theorem Let (x, y, p^N) be a system of parameters in S. Then if $zp^N \in (x, y)$, we have $p^{1/n}z \in (x, y)S^+$ for all positive integers n.

Note: these nth roots of p are all in S^+.

The ring S^+ is Cohen-Macaulay if and only if $zp^N \in (x, y)$ implies $z \in (x, y)$ for all z, or

$$\{z | zp^N \in (x, y)\}/(x, y) = 0.$$

Heitmann's theorem says that $\{z | zp^N \in (x, y)\}/(x, y)$ is almost zero, where $\mathfrak{m}$ is the ideal (p^{1/p^∞}).

Heitmann also showed that this is enough to prove the Direct Summand Conjecture.

Shortly thereafter, Hochster showed that it is also enough to prove the existence of Cohen-Macaulay algebras.

Almost Cohen-Macaulay Algebras

Assume we have a concept of almost zero corresponding to an ideal $\mathfrak{m}$. Let A be a local ring as above and let $x_1, \dots x_d$ be a system of parameters. Let B be an extension of A. We say that $x_1, \dots x_d$ is an *almost regular sequence* in B if $\mathfrak{m}\{z | zx_{i+1} \in (x_1, \dots, x_i))\} \subseteq (x_1, \dots x_i)$ for all $i = 0, \dots d-1$.

We say that B is an *almost Cohen-Macaulay algebra* if

1. Every system of parameters of A is almost regular in B.
2. For a system of parameters $x_1, \dots x_d$ of A, $B/(x_1, \dots x_d)B$ is not almost zero.

The existence of almost Cohen-Macaulay algebras implies many of the homological conjectures.

A Short Digression

There are more general definitions of almost zero and thus of almost Cohen-Macaulay algebras that imply the Direct Summand and other conjectures.

In a paper with A. Singh and Srinivas we asked whether R^+ was almost Cohen-Macaulay in characteristic zero in this sense.

Recently Heitmann and Ma showed that this is true in mixed characteristic (using results of André).

Almost Ring Theory

In this section we discuss almost rings and almost modules. We assume that we have a ring R together with an ideal $\mathfrak{m}$ of R such that $\mathfrak{m}^2 = \mathfrak{m}$. Recall that an R-module is *almost zero* if $\mathfrak{m}M = 0$.

The main reference is Gabber and Ramero, *Almost Ring Theory*.

The main examples we use are (p^{1/p^∞}) or (g^{1/p^∞}) for some nonzerodivisor g, but the only extra condition we put on $\mathfrak{m}$ is that it is flat.
We want to define a quotient category modulo almost zero modules.

This is a classical construction that can be done for any full subcategory of an abelian category with the following property: if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, then M is almost zero if and only if M' and M'' are almost zero.

The subcategory of almost zero modules has this property since $\mathfrak{m}^2 = \mathfrak{m}$.

The construction of the quotient category goes like this. We first define an *almost isomorphism* to be a map with kernel and cokernel almost zero. We then define a map from M to N in the quotient category to be a diagram

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & N \\ {\scriptstyle f}\big\downarrow & & \\ M & & \end{array}$$

where f is an almost isomorphism.

Denote the category of R-modules by Mod_R. The associated quotient category is denoted Mod_R^a. There is a functor $Mod_R \to Mod_R^a$; the image of a module M is denoted M^a.

A map in Mod_R^a as in the diagram above can be thought of as gf^{-1}, and this construction is analogous to localization of rings.

This construction makes a quotient which is an abelian category.

Almost Properties

Almost Flat: An almost module M^a is *almost flat* if $Tor_i(M, N)$ is almost zero for all $i > 0$ and all N.

Almost projective: An almost module M^a is *almost projective* if $Ext^i(M, N)$ is almost zero for all $i > 0$ and all N.

One can define a tensor product in Mod_R^a by letting $M^a \otimes N^a = (M \otimes_R N)^a$. Then M^a is almost flat if and only if the functor $M^a \otimes -$ is exact.

The quotient functor has a right adjoint that sends M^a to $Hom(\mathfrak{m}, M)$, which is denoted M_*.

If S is an R-algebra, S^a can be given the structure of an algebra over R^a.

Almost finitely generated modules

A module M is *almost finitely generated* if for any $g \in \mathfrak{m}$ we can find a finitely generated submodule N such that M/N is annihilated by g.

Suppose M^a is an almost module and for every $g \in \mathfrak{m}$ there exists a diagram

$$M \xrightarrow{\alpha} R^n \xrightarrow{\beta} M$$

with $\beta\alpha$ equal to multiplication by g. Then M is almost finitely generated (since gM is contained in the image of R^n) and almost projective (since multiplication on g on $Ext^i(M, -)$ factors through $Ext^i(R^n, -)$, which is zero for $i > 0$. Thus M is almost finite projective.

Finite Étale Extensions

An extension is *finite étale* if it is finite projective and unramified.

We recall some facts about unramified extensions. We say that an extension $R \to S$ is *unramified* if the module of differentials $\Omega_{S/R}$ is zero. One way of defining $\Omega_{S/R}$ is by taking the multiplication map $S \otimes_R S \to S$; then if I is the kernel of this map, $\Omega_{S/R} \cong I/I^2$. Then S is unramified over R if and only if $I = I^2$, and if S is finite over R, standard commutative algebra implies that there is an element $i \in I$ such that $1 - i$ annihilates I.

Unramified Extensions

Let R, S, I, and i be as above, with S unramified over R, and such that $1 - i$ annihilates I.

1. For all $j \in I, j = ij$.
2. i is idempotent and generates I as an ideal.
3. Let $e = 1 - i$. Then e is an idempotent element that annihilates I.
4. e maps to 1 under the multiplication map.

Conversely, if there exists an element e with the third and fourth properties, S is unramified over R.

The advantage of this is that from the idempotent element one can construct a representation of S as a projective module. Let $e = \sum a_i \otimes b_i \in S \otimes_R S$. We then define

$$\alpha(f) = (Tr_{S/R}(fa_i))$$

and

$$\beta((g_i)) = \sum (g_i b_i).$$

This gives a diagram

$$S \xrightarrow{\quad \alpha \quad} R^n \xrightarrow{\quad \beta \quad} S$$

such that $\beta\alpha$ is the identity on S.

The trace map $Tr_{S/R}$ exists since S is assumed finite projective over R.

The fact that $\beta\alpha$ is the identity on S follows from the fact that the trace of e over S is 1.

We say that S is *almost unramified* over R if there is an idempotent element e in $(S \otimes S)_*$ that annihilates the kernel of the multiplication map and maps to 1 in S_*. S is *almost finite étale* over R if it is almost finite projective and almost unramified.

The Positive Characteristic Case of the Almost Purity Theorem

Assume now that R is a perfect ring of positive characteristic p. Let g be a non-zero-divisor in R.

Theorem

There is an equivalence of categories between the category of finite étale extensions of $R[1/g]$ and the category of almost finite étale extensions of R with respect to (g^{1/p^∞}).

Denote these categories $R[1/g]_{fet}$ and R_{afet} respectively.

The functor $R_{afet} \to R[1/g]_{fet}$ is obtained by inverting g. The functor $R[1/g]_{fet} \to R_{afet}$ is obtained by taking integral closure of R in the extension.

There are two main points:

1. If S is a finite étale extension of $R[1/g]$ the integral closure of R in S is almost finite étale.
2. If T is an almost finite étale extension of R with respect to (g^{1/p^∞}), then the integral closure of R in $T[1/g]$ is almost equal to T.

Proof of (1). Let T be the integral closure of R in S. Let e be an idempotent of $S \otimes_{R[1/g]} S$ that annihilates the kernel of the multiplication map $S \otimes_{R[1/g]} S \to S$ and maps to 1 in S.

Since T is integrally closed in the perfect ring S, T is perfect, and so is $T \otimes_R T$. For some k, $g^{p^k} e$ is in $T \otimes_R T$.

Then $(g^{1/p^n} e)^{p^{n+k}}$ is in $T \otimes T$, so $g^{1/p^n} e$ is in $T \otimes T$.

This then defines a map from the ideal (g^{1/p^∞}) to T, so an element of $(T \otimes T)_*$ with the right properties. Thus T is almost unramified over R.

The element $e_n = g^{1/p^n} e$ satisfies $e_n^2 = g^{1/p^n} e_n$. Using this element in the argument with the trace gives maps

$$T \xrightarrow{\quad \alpha \quad} R^n \xrightarrow{\quad \beta \quad} T$$

whose composition is multiplication by g^{1/p^n} From this we conclude that T is almost finite projective.

Proof of (2). let U be an almost finite étale extension of R and let T be the integral closure of R in $U[1/g]$. Since U is almost finite étale over a perfect ring it is perfect. Let $t \in T$. Then since t is integral, there is a c such that $t^{p^n} \in g^{-c}U$ for all n. Thus $t^{p^n}g^c$ is in U, and since U is perfect, tg^{c/p^n} is in U. Hence t is almost in U.

Conversely, if $u \in U$, then since U is almost finitely generated and contains all powers of u, the set of gu^N is contained in a finitely generated module over R and hence over T. Thus there is a c with $gu^N \in g^{-c}T$ for all N. Thus $g^{c+1}u^{p^n}$ is in T, so $g^{(c+1)/p^n}$ is integral over T so is in T. Hence u is almost in T.

Comments

This theorem can be used to show the existence of almost Cohen-Macaulay algebras in positive characteristic.

Of course, the homological conjectures under discussion were proven long ago in this case, and there are easier ways to construct almost Cohen-Macaulay algebras.

However, the positive characteristic case is used in the theory in mixed characteristic.