PERFECTOID RINGS

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1. Disclaimer

This is the transcription of a lecture given by Wiesława Nizioł for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture introduces the notion of a perfectoid algebra over a perfectoid field, and gives some theorems on tilting equivalences.

2. Preliminaries

Fix a prime $p > 0$.

Definition 2.1. A perfectoid field is a complete nonarchimedean field K with residue characteristic p, with an associated non-discrete rank 1 valuation, such that

$$\mathrm{Frob}_p : K^\circ/p \to K^\circ/p$$

is surjective, where K° denotes the power bounded elements in K. We let $K^{\circ\circ}$ be the maximal ideal in K°.

Remark 2.2. If K has characteristic p, then this is equivalent to saying that K is a complete perfect field.

To a perfectoid field in any characteristic, we associate the "tilt"

$$K^\flat := K^{\circ\flat}[\frac{1}{\pi}]$$

where ϖ is some element in K° such that $|p| \le |\varpi| < 1$. $K^\flat$ is perfectoid with valuation $|x|_{K^\flat} = |x^\sharp|$ induced by the multiplicative "untilt"

$$\lim_{x \mapsto x^p} K^\circ \to K^\circ, x \mapsto x^\sharp,$$

In fact, by Lemma 3.4 in [1], we can assume that ϖ is actually $t^\sharp$ for some $t \in K^\flat$, which, by definition, has the same valuation. One key property that will be used throughout the talk is that there is an isomorphism

$$K^\circ/\varpi \cong K^{\flat\circ}/t.$$

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[1]Typeset by Ashwin Iyengar.

Example 2.3. Let $K = \mathbf{Q}_p(p^{1/p^\infty})$. Then $K^\flat = \mathbf{F}_p[[t^{1/p^\infty}]][1/t]$. Pick $\varpi = p$ and $t = t$. Then

$$K^\circ/p \cong \mathbf{Z}_p[p^{1/p^\infty}]/p \cong \mathbf{F}_p[t^{1/p^\infty}]/t = K^{\flat\circ}/t$$

If in fact char $K = p$ in the first place, then $K \cong K^\flat$.

3. Perfectoid Rings

Definition 3.1. If K is a perfectoid field, then a Banach K-algebra R is called perfectoid if $R^\circ \subseteq R$ is bounded and open, and $\mathrm{Frob} : R^\circ/\varpi \to R^\circ/\varpi$ is surjective.

Then we let Perf_K denote the category of perfectoid K-algebras, whose morphisms are continuous algebra homomorphisms.

Example 3.2.

(1) We may define the perfectoid unit circle by

$$R = K\langle T^{1/p^\infty}\rangle := \widehat{\bigcup_n K^\circ[T^{1/p^\infty}]}[1/\varpi]$$

(2) One can also define an integral version $K^\circ\langle T^{1/p^\infty}\rangle$.

(3) In characteristic p, if A is a Banach K-algebra with A° bounded and open, then A is perfectoid $\iff$ A is perfect.

The category Perf_K enjoys some nice stability properties, namely

(1) If $B, C \in \mathsf{Perf}_K$, then $B\widehat{\otimes}_K C \in \mathsf{Perf}_K$, and

$$(B\widehat{\otimes}_K C)^\circ \cong_a B^\circ\widehat{\otimes}_{K^\circ} C^\circ$$

(the a denotes an almost isomorphism).

(2) If B is finite étale over $A \in \mathsf{Perf}_K$, then B is perfectoid and B° is almost étale over A°.

4. Tilting Equivalence

The first theorem, due to Scholze, relates K and $K^\flat$ as follows:

Theorem 4.1 (Scholze)**.** *There is an equivalence of categories*

$$\mathsf{Perf}_K \cong \mathsf{Perf}_{K^\flat}.$$

In this section we will describe this equivalence and give a sketch of the proof.

Proof. We begin with some definitions:

Definition 4.2. A $K^{\circ a}$-algebra A is perfectoid if

(1) A is ϖ-adically complete and (almost) flat over $K^{\circ a}$, and

(2) $K^\circ/\varpi \to A/\varpi$ is relatively perfect, i.e.

$$\mathrm{Frob}_p : A/\varpi^{1/p} \to A/\varpi$$

is an isomorphism.

This defines a category $\mathsf{Perf}_{K^{\circ a}}$ for the almost K-algebra $K^{\circ a}$.

Definition 4.3. A $K^{\circ a}/\varpi$-algebra A is perfectoid if

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This defines a category $\mathsf{Perf}_{K^{\circ a}/\varpi}$ for the almost K-algebra $K^{\circ a}/\varpi$.

Step 1: Tilting from char 0 to char p

There exist canonical equivalences

$$\mathsf{Perf}_K \cong \mathsf{Perf}_{K^{\circ a}} \cong \mathsf{Perf}_{K^{\circ a}/\varpi}.$$

The first equivalence is explained as follows: given a perfectoid K-algebra M, one can associate to it the "almost" $K^{\circ a}$-module $M^{\circ a}$. In the reverse direction, one associates to a $K^{\circ a}$-module A the perfectoid K-algebra $A_*[1/\varpi]$, where

$$A_* = \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$$

is the right adjoint to $M \mapsto M^a$. This is justified by the following lemma:

Lemma 4.4 ([1] Lemma 5.6)**.** *If $R = A_*[1/\varpi]$, with Banach K-algebra structure given by making A_* an open and bounded subring, then $R^\circ = A_*$ and $R \in \mathsf{Perf}_K$.*

The second equivalence uses almost deformation theory, which is controlled by the cotangent complex. We describe the usual cotangent complex, then mention the almost cotangent complex.

Definition 4.5. If $f : A \to B$ is a ring homomorphism, then the cotangent complex $L_{B/A}$ is an object in $D^{\leq 0}(\mathsf{Mod}_B)$, the bounded-above derived category of B-modules. Specifically, we let $L_{B/A}$ be the complex associated to

$$\Omega^1_{P_\bullet} \otimes_{P_\bullet} B,$$

where $P_\bullet \to B$ is a simplicial resolution of B by polynomial A-algebras, and the associated complex is the complex whose differentials are alternating sums of the face maps.

We list some properties of this construction.

(1) If $A \to B$ is smooth, then $L_{B/A} = \Omega^1_{B/A}[0]$, i.e. the complex concentrated at 0 with the object $\Omega^1_{B/A}$.

(2) If $A \to B$ is étale, then $L_{B/A} = 0$.

(3) $L_{B/A}$ controls deformations, in a certain sense.

(4) One can define an extension of this notion to the almost category.

Example 4.6. If $A = \mathbf{F}_p$ and B is a perfect $\mathbf{F}_p$-algebra, then $L_{B/A} = 0$. This is because $\mathrm{Frob}_p : B \to B$ is an isomorphism so $d\varphi : L_{B/A} \to L_{B/A}$ is an isomorphism, but $d\varphi = 0$ because "everything is a p-power", in some sense.

More generally, we have the following:

Lemma 4.7. *If A is a perfectoid $K^{\circ a}/\varpi$-algebra, then*

$$L_{A/(K^{\circ a}/\varpi)} = 0.$$

One can show that this implies that

$$\mathsf{Perf}_{K^{\circ a}/\varpi} \cong \mathsf{Perf}_{K^{\circ a}/\varpi^n}$$

for all n, so one can then pass to the limit $K^{\circ a} = \varprojlim_n K^{\circ a}/\varpi^n$, to deduce the result. This proves that

$$\mathsf{Perf}_{K^{\circ a}/\varpi} \cong \mathsf{Perf}_{K^{\circ a}}.$$

In characteristic p, this is simpler: given $A \in \mathsf{Perf}_{K^{\flat\circ a}/t}$, we take

$$\widetilde{A} = \lim_{x \mapsto x^p} A \in \mathsf{Perf}_{K^{\flat\circ a}},$$

where the limit is actually an "almost" limit, taken in the almost category of $K^{\flat\circ a}$-modules.

Step 2 We have a chain of equivalences

$$\begin{array}{ccccc}
\mathsf{Perf}_K & \xrightarrow{\sim} & \mathsf{Perf}_{K^{\circ a}} & \xrightarrow{\sim} & \mathsf{Perf}_{K^{\circ a}/\varpi} \\
 & & & & \| \\
\mathsf{Perf}_{K^\flat} & \xrightarrow{\sim} & \mathsf{Perf}_{K^{\flat\circ a}} & \xrightarrow{\sim} & \mathsf{Perf}_{K^{\flat\circ a}/\varpi}
\end{array}$$

The horizontal equivalences are as described above: in particular, the first uses the adjoint construction $M \mapsto M_*$, and the second uses almost deformation theory.

Essentially, we have shown that given $R \in \mathsf{Perf}_K$, there exists a unique (up to isomorphism) $S \in \mathsf{Perf}_{K^\flat}$ such that

$$R^{\circ a}/\varpi \cong S^{\circ a}/t.$$

Thus, we may call $R^\flat := S$, and we say that R is the "until" of S, i.e. $R = S^\sharp$. This concludes the proof.

□

5. Finite Étale Equivalence

We now state our second theorem, which relates finite étale covers of a perfectoid field to those of their tilts.

Theorem 5.1 (Scholze). *If R is perfectoid over K, then there exists an equivalence*

$$R_{f\acute{e}t} \cong R^\flat_{f\acute{e}t},$$

where $R_{f\acute{e}t}$ is the category of finite étale covers of R.

Remark 5.2. Fontaine and Wintenberger proved this in the case $R = K$ for certain K. In particular, they showed that $\mathrm{Gal}_K = \mathrm{Gal}_{K^\flat}$ in certain cases.

Proof. We give a sketch of the proof. As before, there are equivalences

$$\begin{array}{ccccc} R_{\text{fét}} & \xrightarrow{\sim} & R^{\circ a}_{\text{fét}} & \xrightarrow{\sim} & (R^{\circ a}/\varpi)_{\text{fét}} \\ & & & & \| \\ R^\flat_{\text{fét}} & \xrightarrow{\sim} & R^{\flat\circ a}_{\text{fét}} & \xrightarrow{\sim} & (R^{\flat\circ a}/\varpi)_{\text{fét}} \end{array}$$

This time, the first horizontal arrows use the almost purity theorem. On the bottom, the theorem is for characteristic p, and on the top, in characteristic 0. The second horizontal arrows again use almost deformation theory, although we won't go into the details. □

References

[1] P. Scholze, *Perfectoid Spaces*, Publications mathématiques de l'IHÉS 116(1), 2012, 245–313.