A BRIEF INTODUCTION TO ADIC SPACES

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1. Valuation spectra and Huber/Tate rings

1.1. **Introduction.** Although we begin the oral lectures with a crash course on some basic highlights from rigid-analytic geometry in the sense of Tate, some awareness of those ideas is taken as known for the purpose of reading these written notes that accompany those lectures. The introductory survey [C2] provides (much more than) enough about such background from scratch (with specific references to the literature for further details).

Let k be a non-archimedean field (i.e., a field complete with respect to a non-trivial non-archimedean absolute value $|\cdot| : k \to \mathbf{R}_{\geq 0}$). For a k-affinoid algebra A, on the set $\mathrm{Sp}(A) = \mathrm{MaxSpec}(A)$ Tate defined a notion of "admissible open" subset and "admissible cover" of such a subset in a manner that forces a compactness property. This Grothendieck topology restores a type of "local connectedness" that is not available in the traditional theory of analytic manifolds over non-archimedean fields as in [Se, Part II, Ch. III]. We refer the reader to [C3, 1.2.6-1.2.9, 1.3] for an instructive analogy of Tate's "admissibility" idea in the context of usual Euclidean geometry to use a totally disconnected space to probe the topology of a richer ambient space.

Tate's mild Grothendieck topology defines a category $\mathrm{Shv}(A)$ of sheaves of sets (the "Tate topos") in which we have a good theory of coherent modules over a certain structure sheaf $\mathscr{O}_A$ (whose existence is also a deep result of Tate). But there are deficiencies:

- (i) For an extension K/k of non-archimedean fields, we have a map $A \to A_K := K\widehat{\otimes}_k A$ from a k-affinoid algebra to a K-affinoid algebra but if $[K : k]$ is infinite then typically there is no evident map $\mathrm{Sp}(A_K) \to \mathrm{Sp}(A)$. (The same issue comes up for schemes of finite type over fields if we only use closed points.)
- (ii) There are "not enough points" in $\mathrm{Sp}(A)$ in the sense that the stalk functors $\mathscr{F} \rightsquigarrow \mathscr{F}_x = \varinjlim_{x\in U} \mathscr{F}(U)$ for $x \in \mathrm{Sp}(A)$ are insufficient to detect if an abelian sheaf is nonzero, etc.
- (iii) Admissibility is a tremendous pain when trying to carry out global constructions such as moduli spaces not arising from algebraic geometry (e.g., representability of rigid-analytic Picard and Hilbert functors in the proper setting without the presence of a relatively ample line bundle, by trying to adapt M. Artin's manifestly "pointwise" methods).

Berkovich overcame some of these defects by introducing an enhanced space $\mathcal{M}(A)$ that encodes bounded k-algebra maps $A \to k'$ to "all" non-archimedean fields k'/k. However, his global spaces are not full subcategories of the category of locally ringed spaces. Huber's solution involves an enhanced space $\mathrm{Spa}(A)$ that (roughly speaking) encodes all "continuous" k-algebra maps $A \to K$ to valued fields K whose value group $\Gamma \supset |k^\times|$ is an arbitrary totally ordered abelian group (not necessarily a subgroup of $\mathbf{R}_{>0}$). This has some attractive features:

1. This is a locally ringed space (no "admissibility" or G-topology), and the underlying topological space is *spectral* (see Definition 1.15; this permits arguments with generic points and specialization as in algebraic geometry).

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(2) It allows a wide class of "non-archimedean" rings in a *useful* way (for which real theorems can be proved), so no ground field is required and one unifies rigid-analytic spaces over k and appropriate formal schemes over $\mathscr{O}_k$ as part of a common geometric category.
(3) Although $\mathrm{Spa}(A)$ typically has many new closed points (unlike the passage from classical algebraic varieties to the associated schemes), there are *lots* of rank-1 points (some closed, some not closed), so non-archimedean fields continue to play an essential role in the general theory. The "higher rank" points can also be closed. See Example 2.26 for an explicit illustration of the various types of non-classical points.
(4) The category $\mathrm{Shv}(\mathrm{Spa}(A))$ of sheaves of sets on the spectral topological space $\mathrm{Spa}(A)$ coincides with Tate's original topos $\mathrm{Shv}(A)$ of sheaves of sets for the Tate topology on $\mathrm{Sp}(A)$; it then follows from general facts about spectral spaces that stalks at points of $\mathrm{Spa}(A)$ constitutes all "points" of the Tate topos. This theorem of Huber is never used in what follows, but is quite beautiful and is perhaps psychologically reassuring.

Essentally everything we discuss in these notes beyond some basic facts about valuation rings is due to Huber. For the reader's convenience we will generally refer to seminar notes [C3] for omitted details and proofs related to Huber's work on adic spaces, and the original references to Huber's papers are given in [C3].

1.2. **Review of valuation rings.** We shall consider valuation rings with valuation written in multiplicative notation (for harmony with conventions for non-archimedean fields).

Definition 1.3. A *valuation ring* is a domain R with fraction field K such that for each $x \in K^\times$ either $x \in R$ or $1/x \in R$. The *value group* is $\Gamma_R = K^\times/R^\times$.

We make Γ_R into a totally ordered abelian group by defining $a \bmod R^\times \leq b \bmod R^\times$ (for $a, b \in K^\times$) to mean $a/b \in R$. The natural map $v : K \to \Gamma_R \cup \{0\}$ (sending 0 to 0) then satisfies the following properties: $v(x) = 0$ if and only if $x = 0$, $v(K^\times) = \Gamma_R$, $v(xy) = v(x)v(y)$ (where $0 \cdot \gamma := 0$ for all $\gamma \in \Gamma_R$), and $v(x + y) \leq \max(v(x), v(y))$.

Remark 1.4. We allow the possibility $\Gamma_R = 1$ (i.e., $R = K$), and we emphasize that the value group is an *abstract* totally ordered abelian group; it is not specified inside $\mathbf{R}_{>0}$ (in contrast with Berkovich spaces, for which such an embedding is part of the data; in general such an embedding might not exist at all).

We say that R (or Γ_R) has *rank* 1 (or more accurately, rank ≤ 1 by allowing the case $\Gamma_R = 1$) when there exists a subgroup inclusion $\Gamma_R \hookrightarrow \mathbf{R}_{>0}$ that is order-preserving in both directions.

Exercise 1.5. For a totally ordered abelian group $\Gamma \neq 1$, show that Γ has rank 1 if and only if for all $\gamma < 1$ in Γ, the powers $\{\gamma^n\}_{n>0}$ constitute a cofinal subset of Γ.

Example 1.6. Here is an example of a rank-2 valuation ring that is worth studying very carefully, as it will arise repeatedly later on as a prototype for many general situations. Let R be a valuation ring with residue field κ, and suppose on κ there is specified a valuation with valuation ring $\overline{R} \subset \kappa$. For instance, we could have $R = k(\!(u)\!)[\![t]\!]$ with the t-adic valuation, $\kappa = k(\!(u)\!)$ on which there is specified the u-adic valuation whose valuation ring is $k[\![u]\!]$.

Define the subring $R' \subset R$ to be the Cartesian product

$$R' = \overline{R} \times_\kappa R = \{x \in R \,|\, x \bmod \mathfrak{m}_R \in \overline{R}\}.$$

Note that $\mathfrak{m}_R \subset R'$ with $R'/\mathfrak{m}_R = \overline{R}$, so in particular R' is a valuation ring (check!).

Provided that R and $\overline{R}$ are not fields, we claim that such R' is *never* a rank-1 valuation ring. Indeed, we can choose a nonzero $t \in \mathfrak{m}_R$ and an element $u \in R$ whose image in κ is a nonzero element of $\mathfrak{m}_{\overline{R}}$, so $t/u^n \in R'$ for all $n > 0$ (why?). Letting v' be the valuation on R', it follows that $v'(t) \le v'(u)^n$ for all $n > 0$ yet $v'(u) < 1$. Thus, $\{v'(u)^n\}_{n>0}$ is not cofinal in $\Gamma_{R'}$, so R' is not rank-1.

In the toy case $R = k((u))[\![t]\!]$ as considered above, we have

$$\Gamma_{R'} \simeq a^{\mathbf{Z}} \times b^{\mathbf{Z}}$$

with the lexicographical ordering where $0 < a, b < 1$ (so $v'(t) = (a, 1)$, $v'(u) = (1, b)$).

For any valued field (K, v) with value group Γ, we topologize K using the base of opens

$$B(a, \gamma) = \{x \in K \mid v(x - a) < \gamma\}$$

for $a \in K$ and $\gamma \in \Gamma$. (We do *not* use the condition $v(x - a) \le \gamma$, since in the case $\Gamma = \{1\}$ we want to get the discrete rather than indiscrete topology.)

Here is a very important and perhaps initially surprising fact:

Exercise 1.7. Show that the topology on $k((u))((t))$ arising from the rank-1 valuation $v = \mathrm{ord}_t$ coincides with the topology defined by the rank-2 valuation v'.

That a higher-rank valuation on a field can define the *same valuation topology* as a rank-1 valuation on that field is initially disorienting but is a pervasive phenomenon in what follows. It incorporates the possibility on an element x that $\{x^n\}_{n>0}$ may be "bounded" even if $v(x) > 1$; e.g., $1 < v'(1/u)^n \le v'(1/t)$ for all $n > 0$ in our toy example of a rank-2 valued field. Here is a characteristization of such valuations (sometimes called *mircobial*):

Proposition 1.8. *If v is a nontrivial valuation on a field K and R is its valuation ring, then the following are equivalent:*

- (i) *The v-topology on K coincides with that of a rank-1 valuation on K.*
- (ii) *There exists a nonzero topologically nilpotent element in K.*
- (iii) *The ring R admits a height-1 prime ideal.*

The proof of this result requires some effort; see [C3, 9.1.3] for the details. In practice condition (ii) will be easy to verify, and the implication "(ii) $\Rightarrow$ (i)" will have useful consequences for the abundance of "rank-1 points" in certain adic spaces.

Definition 1.9. A *valuation* on a commutative ring A is a pair $(\mathfrak{p}, R)$ where $\mathfrak{p} \in \mathrm{Spec}(A)$ and R is a valuation ring on the residue field $\kappa(\mathfrak{p})$; i.e., we are given a function $v : A \to \Gamma \cup \{0\}$ for a totally ordered abelian group Γ such that $v(0) = 0$, $v(1) = 1$, $v(xy) = v(x)v(y)$, $v(x+y) \le \max(v(x), v(y))$, $\mathfrak{p} := v^{-1}(0)$ is prime, and Γ is generated by $v(A - \mathfrak{p})$.

Note that the pairs $(\mathfrak{p}, \Gamma)$ with $\Gamma = 1$ correspond to the trivial valuations on the residue fields of A at its prime ideals. Also, the definition of a valuation is set up to make the "value group" intrinsic and minimal, thereby avoiding annoying issues of "equivalence" of valuations.

1.10. **Valuation spectra.** Let A be a commutative ring.

Definition 1.11. The *valuation spectrum* $X = \mathrm{Spv}(A)$ is the set of valuations on A, equipped with a topology for which a base of opens is given by the subsets

$$X\left(\frac{f_1, \dots, f_n}{s}\right) := \{v \in X \mid v(f_1), \dots, v(f_n) \le v(s) \ne 0\}.$$

(Note that we manually insert the condition $v(s) \ne 0$. Informally, the requirement on v is that each fraction f_j/s is v-integral.)

For a point $x \in X$ corresponding to a valuation $v : A \to \Gamma \cup \{0\}$ on A, we often write $|f(x)|$ rather than $v(f)$ (inspired by the classical rigid-analytic case) even though typically Γ is not of rank 1 (and even when it is of rank 1, we do *not* choose an embedding of it into $\mathbf{R}_{>0}$).

Remark 1.12. Beware that since typically s is not a unit in the subring $A[f_1/s, \dots, f_n/s] \subset A[1/s]$, so it is typically killed by some valuations on this subring, the subset $X(\{f_1, \dots f_n\}/s) \subset X$ is usually *not* the image of $\mathrm{Spv}(A[f_1/s, \dots, f_n/s])$. In particular, an assertion of quasi-compactness for valuation spectra of arbitrary rings does not formally imply the same for such opens defining the topology (though it will turn out that such subsets *are* quasi-compact: see Theorem 1.16). This situation thereby differs from that of affine schemes, for which the basic affine opens are topologically spectra of rings.

Example 1.13. If $A = K$ is a field then $\mathrm{Spv}(A)$ recovers as a topological space the Riemann–Zariski space $\mathrm{RZ}(K)$ with its base of open sets $\mathrm{RZ}(K, \{a_1, \dots, a_n\})$ consisting of those valuations v on K whose associated valuation ring R_v contains each of finitely many specified elements $a_1, \dots, a_n$. In particular, the specialization relation $v \in \overline{\{w\}}$ in $\mathrm{Spv}(K)$ says exactly $R_v \subset R_w$ inside K.

Exercise 1.14. Prove that the natural map $\mathrm{Spv}(A) \to \mathrm{Spec}(A)$ assigning to each $v \in \mathrm{Spv}(A)$ its *support* $\mathfrak{p}_v := v^{-1}(0)$ is continuous, and that the fiber over $\mathfrak{p}$ is identified as a topological space (not just as a set!) with $\mathrm{RZ}(\kappa(\mathfrak{p}))$.

Using that this fiber has the subspace topology, and that specialization relations among points can be expressed in terms of open subsets (whereas the formation of closure doesn't generally commute with intersecting with a fiber!), show that for two points $v, w \in \mathrm{Spv}(A)$ with the same support $\mathfrak{p}$, we have $v \in \overline{\{w\}}$ if and only if $R_v \subset R_w$ as subrings of $\kappa(\mathfrak{p})$.

Visualizing $\mathrm{Spv}(A)$ as lying "over" $\mathrm{Spec}(A)$, we refer to specialization relations in a fiber as *vertical specialization* (or *vertical generization*). There is an entirely different specialization construction (that we will not be discussing here) called *horizontal specialization* concerning v and w in distinct fibers. It is very important for the proofs of some fundamental results on valuation spectra that general specialization relations in $\mathrm{Spv}(A)$ can be systematically built from iterating the operations of vertical and horizontal specialization in a controlled manner. This provides the only tangible way to visualize what specialization looks like inside $\mathrm{Spv}(A)$ (substituting for how we visualize specialization among points in $\mathrm{Spec}(A)$). This is discussed in depth in [C3, Lecture 4].

The following notion is due to Hochster, motivated by topological properties of spectra of rings.

Definition 1.15. A topological space X is *spectral* if it is quasi-compact, quasi-separated (i.e., the intersection of any two quasi-compact open subsets is quasi-compact), sober (i.e., each irreducible closed subset has a unique generic point), and the quasi-compact open subsets constitute a base for the topology.

Whereas the spectrality of $\mathrm{Spec}(A)$ for any commutative ring A is elementary, the analogue for valuation spectra lies deeper:

Theorem 1.16 (Huber). *The space* $\mathrm{Spv}(A)$ *is spectral and each open subset* $X(\{f_1, \dots, f_n\}/s)$ *is quasi-compact.*

Beware that the quasi-compactness of the subsets $X(\{f_1, \dots, f_n\}/s)$ is *not* a consequence of quasi-compactness of valuation spectra, due to the phenomenon in Remark 1.12. The proof of Theorem 1.16 is addressed in [C3, 3.4], so here we now just sketch the idea for the proof of the quasi-compactness assertions. (The sobriety property is *much harder* to prove.)

One can express the axioms on valuations in terms of properties satisfied by the condition "$v(a) \le v(s) \neq 0$" for varying $(a, s) \in A \times A$, thereby defining a special class of relations $\mathscr{R}$ on

$A \times A$. Viewing such an $\mathscr{R}$ as a point in the power set $\wp(A \times A) = \{0,1\}^{A\times A}$, when this power set is made into a compact Hausdorff space (via the product topology for the discrete topology on $\{0,1\}$) it turns out that the set (ignoring its topology!) $X = \mathrm{Spv}(A)$ is a *closed* subset of $\wp(A \times A)$ in which its subsets $X(\{f_1,\dots,f_n\}/s)$ are clopen for that ambient compact Hausdorff topology. From this one immediately gets the asserted quasi-compactness conditions for the original topology on X (which has nothing to do with its topology inherited from $\wp(A \times A)$!).

Definition 1.17. A topological ring A is *non-archimedean* is 0 admits a base of open neighborhoods that are additive subgroups.

For later considerations with tensor products and localizations we do *not* impose any Hausdorff (or completeness) conditions in the preceding definition. Next, we want to define "continuity" for a valuation $v : A \to \Gamma \cup \{0\}$ when A is a topological ring that is non-archimedean.

Definition 1.18. Let A be a non-archimedean ring. A valuation $v : A \to \Gamma \cup \{0\}$ is *continuous* if for each $a \in A$ and $\gamma \in \Gamma$ the subset

$$\{x \in A \,|\, v(x-a) < \gamma\}$$

is open. (It is equivalent to consider just $a = 0$, since A is a topological group when considered additively.)

Denote by $\mathrm{Cont}(A) \subset \mathrm{Spv}(A)$ the subset of such continuous v, and equip this subset with the subspace topology.

Whereas Theorem 1.16 is for arbitrary commutative rings, what we really need is an analogous such result for $\mathrm{Cont}(A)$ when A belongs to a suitable class of non-archimedean rings. We now turn to the definition of that class of rings.

1.19. **Huber rings and Tate rings.**

Definition 1.20. A topological ring A is *Huber* (originally: "f-adic", where "f" refers to "finite") if there exists an open subring $A_0 \subset A$ (called a "ring of definition") on which the topology is I-adic for a finitely generated ideal I of A_0.

We do *not* require that A_0 is I-adically separated or complete. (This is technically convenient for later considerations with tensor products and rings of fractions.) Note that from the definition it follows that any Huber ring is non-archimedean.

Here are some examples of Huber rings (the first and third of which unify rigid-analytic geometry and noetherian formal schemes into a common framework):

(1) For a k-affinoid algebra $A \simeq k\langle t_1,\dots,t_n\rangle/J$, we can take A_0 to be the image of $\mathscr{O}_k\langle t_1,\dots,t_n\rangle$ and $I = \varpi A_0$ where $\varpi \in k^\times$ is a pseudo-uniformizer: $0 < |\varpi| < 1$.
(2) As a very special case, we can take A to be k, $A_0 = \mathscr{O}_k$, and $I = (\varpi)$; in general we *cannot* take I to be $\mathfrak{m}_{\mathscr{O}_k}$ since for algebraically closed k this maximal ideal is not finitely generated and is even equal to its own square (so its adic topology on $\mathscr{O}_k$ does not give the valuation topology).
(3) $A = A_0$ a noetherian ring with topology defined by an ideal.
(4) If B is a Huber ring and B_0 is a ring of definition with $I_0 \subset B_0$ a finitely generated ideal defining its topology then we get another Huber ring via the relative Tate algebras

$$A = B\langle t_1,\dots,t_n\rangle = \{\sum b_J t^J \,|\, b_J \to 0 \text{ as } \|J\| \to \infty\}$$

with ring of definition A_0 gives by the I_0-adic completion $B_0\{t_1,\dots,t_n\}$ of $B_0[t_1,\dots,t_n]$.

We emphasize that the choice of the pair (A_0, I) is very flexible, and is not special in any way at all. (See [C3, 5.4.10, 5.4.13] for illustrations of this flexibility.)

Definition 1.21. A topological ring A is *Tate* if it is Huber and contains a topologically nilpotent unit ϖ (called a *pseudo-uniformizer*).

Obviously if $A \to B$ is a continuous map of Huber rings and A is Tate then so is B. For any Tate ring A and ring of definition A_0, by replacing ϖ with ϖ^N for large enough N we can arrange that $\varpi \in A_0$. It is then easy to check that $A = A_0[1/\varpi]$. Typically in non-Tate cases one cannot recover the topology on A from the one on A_0 by a simple localization operation.

Informally, Tate rings permit versions of arguments for Banach spaces whereas general Huber rings do not. For example, we will see next that "boundedness" can be defined in the setting of Huber rings but that a continuous map between Huber rings can *fail* to carry bounded sets to bounded sets (whereas this works well for Tate rings).

2. Rings of fractions, adic Nullstellensatz, and rational domains

2.1. **Boundedness.** Let A be a non-archimedean ring (not necessarily Hausdorff nor complete). For any two subsets $S, S' \subset A$, we define $S \cdot S'$ to be the set of finite sums of products ss' for $s \in S$ and $s' \in S'$.

Definition 2.2. A subset $\Sigma \subset A$ is *bounded* if for all open neighborhoods U of 0 there exists an open neighborhood V of 0 such that $V \cdot \Sigma \subset U$. (Since one can restrict to a cofinal system of such U that are additive subgroups, it is the same to consider just individual products $v \cdot \sigma$ rather than finite sums of such.)

It is a simple exercise to check that if Σ_1, Σ_2 are bounded subsets of A then so is $\Sigma_1 \cdot \Sigma_2$ (the set of *finite sums* of pairwise products). We say that Σ is *power-bounded* if the set $\{s_1 \cdots s_n \mid s_j \in \Sigma\}$ of *all* products of finitely many elements of Σ is bounded in A; in the special case of a singleton set $\Sigma = \{a\}$ this is the property that $\{a, a^2, \dots\}$ is bounded, in which case we say that a is *power-bounded*. Using the binomial theorem, it is elementary to check that the set

$$A^0 := \{a \in A \mid a \text{ is power-bounded}\}$$

is a subring of A.

Example 2.3. If A is a reduced k-affinoid algebra then A^0 is even *bounded* in A because $\|\cdot\|_{\text{sup}}$ is a norm that defines the topology on A [BGR, 6.2.4/1].

Example 2.4. If $A = k[\varepsilon]$ is the algebra of dual numbers over k then $A^0 = \mathscr{O}_k + k\varepsilon$ is not bounded. Thus, reducedness is essential to the boundedness of ring of power-bounded elements in the preceding example.

Example 2.5. Let (K, v) be a valued field whose valuation topology coincides with that defined by a rank-1 valuation w on K; we have seen examples in which v is a higher-rank valuation. In such cases K^0 defined purely in terms of the common topology of v and w coincides with the valuation ring of w and *not* the valuation ring for v (except when $v = w$).

Now assume that A is a Huber ring, and let (A_0, I) be a *couple of definition* (i.e., a ring of definition and a finitely generated ideal of A_0 defining its topology). The proof of the following lemma is left as an exercise:

Lemma 2.6. *All open subrings of A are Huber, and an open subring of A is a ring of definition if and only if it is bounded.*

Arguing as for Banach spaces, if $f : A \to B$ is a continuous map between Tate rings then f carries bounded sets onto bounded sets. This fails for “Tate” relaxed to “Huber” (e.g., take B to be a non-archimedean field and A the same underlying field with the discrete topology), but remains correct when f is either a surjective topological quotient map or expresses B as the *completion* of A (a notion to be defined shortly).

2.7. **Operations on Huber rings.** Tensor products can be considered for Huber rings, but this involves some topological subtleties (related to the fact that the “generic fiber” of $\mathrm{Spf}(\mathbf{Z}_p[\![T]\!])$ over $\mathrm{Sp}(\mathbf{Q}_p)$ should be the open unit disc that is *not affinoid*). Thus, we will discuss tensor products here only in the Tate case, as in the following rather easy result:

Proposition 2.8. *Let C be a Tate ring and $\varpi \in C$ a pseudo-uniformizer (i.e., topologically nilpotent unit). Let $C \rightrightarrows A, B$ be continuous maps to Huber rings (so A and B are Tate), and let C_0, A_0, B_0 be compatible rings of definition such that $\varpi \in C_0$ (as can always be arranged). Then $A \otimes_C B$ is a Tate ring when using $\mathrm{im}(A_0 \times_{C_0} B_0)$ as a ring of definition equipped with its ϖ-adic topology, and it satisfies the expected universal property relative to continuous maps into non-archimedean target rings.*

The universal property in the preceding result shows that the topological ring structure on $A \otimes_C B$ is independent of the auxiliary choices of ϖ and compatible rings of definition.

Next, we turn to an important construction involving rings of fractions for Huber rings, as a warm-up to the construction of “rational domains” for Huber rings (generalizing the notion of rational domain in rigid-analytic geometry). Let A be a Huber ring, and choose a finite subset $T := \{f_1, \ldots, f_n\} \subset A$ generating an *open* ideal $T \cdot A \subset A$. (If A is Tate then the only open ideal is the unit ideal – why? – so this says $T \cdot A = (1)$ in such cases.) For $s \in A$, define $A(T/s)$ to be the ring $A[1/s]$ equipped with ring of definition $A_0[T/s] := A_0[f_1/s, \ldots, f_n/s] \subset A[1/s]$ equipped with the I-adic topology, where (A_0, I) is a couple of definition for A. The following result shows that $A(T/s)$ as a topological ring structure on $A[1/s]$ is independent of the choice of (A_0, I) as just used:

Theorem 2.9. *The topological ring $A(T/s)$ is Huber and has the universal property that the natural continuous map $A \to A(T/s)$ is initial among continuous maps $f : A \to B$ to non-archimedean rings such that $f(s) \in B^\times$ and $f(t)/f(s) \in B^0$ for all $t \in T$.*

Remark 2.10. Despite the simple formulation of the preceding theorem, the proof requires some real work: it uses crucially that openness of $T \cdot A$ implies openness of $T \cdot U$ for any open neighborhood U of 0, as shown in [C3, 6.3.5], and uses topological considerations with relative Tate algebras that we prefer not to get into here (see [C3, §6.3] for further details).

The last general operation we introduce, albeit briefly, is completion. Letting (A_0, I) be a couple of definition for a Huber ring A, the *completion* is

$$\widehat{A} = \varprojlim A/I^n$$

(where the quotients A/I^n are *not rings* since I^n is merely an additive subgroup of A and generally not an ideal of A, but the inverse limit nonetheless has a well-defined ring structure in an evident manner since $I^n \to 0$ in A). This is a Hausdorff topological ring in which the I-adic completion $\widehat{A_0}$ is an open subring having the I-adic topology that is separated and complete (see the Stacks Project for good properties of completion of general rings with respect to finitely generated ideals).

This makes $\widehat{A}$ a Huber ring characterized by a universal property (so it is independent of the choice of (A_0, I)), and $\widehat{A_0} \otimes_{A_0} A \to \widehat{A}$ is an isomorphism (see [C3, §5.5] for further details). In particular, combining our "rational domain" operation on Huber rings with the completion as just discussed, we see that

$$A\langle T/s\rangle := A(T/s)^{\wedge}$$

is a Huber ring with a universal property similar to that of rational domains in rigid-analytic geometry.

2.11. **Spa revisited.** Let A be a Huber ring, and $A^{00} \subset A^0$ the open ideal of topological nilpotents in A^0. For any subset $\Sigma \subset A$ define

$$\mathrm{Spa}(A, \Sigma) := \{v \in \mathrm{Cont}(A) \,|\, v(a) \leq 1 \text{ for all } a \in \Sigma\} = \mathrm{Spa}(A, \mathbf{Z}[\Sigma + A^{00}]^{\sim});$$

where $\mathbf{Z}[\Sigma + A^{00}]^{\sim}$ denotes the integral closure in A of the open subring $\mathbf{Z}[\Sigma + A^{00}]$. Note that this integral closure is an open and integrally closed subring of A. (Here and below we slightly abuse terminology and say "integrally closed subring of A" to mean a subring R of A such that all elements of A integral over R belong to R; this is not to be confused with any more intrinsic notion of "integral closedness" for R without reference to A, as one might imagine at least when A is a domain.)

Theorem 2.12. *The space* $\mathrm{Cont}(A)$ *is spectral for the subspace topology from* $\mathrm{Spv}(A)$*.*

See [C3, 8.4.1, §9] for some discussion of the proof of this result. It rests on a *lot* of study of specialization in $\mathrm{Spv}(A)$ and the (highly non-obvious) "nearly algebraic" formula

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A, A^{00} \cdot A) \,|\, v(a) < 1 \text{ for all } a \in A^{00}\} \tag{1}$$

proved in [C3, 9.3.1] (the right side involves the topology on A *solely* through the specification of the ideal A^{00} via a technical but purely algebraic construction $\mathrm{Spv}(R, J) \subset \mathrm{Spv}(R)$ defined in [C3, §9.2] for any commutative ring R and ideal J whose radical coincides with the radical of a finitely generated ideal, using convexity conditions on valuations).

A consequence of the method of proof of Theorem 2.12 is:

Corollary 2.13. *For a Huber ring A, the space* $\mathrm{Spa}(A, \Sigma)$ *is spectral for any subset $\Sigma \subset A$. In particular,* $\mathrm{Spa}(A, A^+)$ *is spectral for any open and integrally closed subring of A.*

The formula (1) also underlies the proof of the following fact in [C3, 11.4.1] governing the structure of rational domains in the general theory:

Theorem 2.14. *Let A^+ be an open and integrally closed subring of a Huber ring A, and let X denote the spectral space* $\mathrm{Spa}(A, A^+)$*. For a finite subset $T \subset A$ generating an open ideal in A and any $s \in A$, the open subset $X(T/s) = \{v \in X \,|\, , v(t) \leq v(s) \neq 0$ for all $t \in T\}$ is quasi-compact and the natural map*

$$\mathrm{Spa}(A(T/s), A^+[T/s]) \to X \tag{2}$$

is a homeomorphism onto $X(T/s)$. In particular, each such $X(T/s)$ is spectral.

Moreover, a subset $W \subset X(T/s)$ is a "rational domain" in X with respect to A (i.e., $W = X(T'/s')$ for $s' \in A$ and finite $T' \subset A'$ generating an open ideal) if and only if it is a "rational domain" in $X(T/s)$ with respect to $A(T/s)$.

Note the contrast of the image of (2) with the failure of the analogue in Remark 1.12 for valuation spectra and general pairs (T, s) (in the absence of any "open ideal" condition on T).

Remark 2.15. Beware that the quasi-compactness of

$$(\mathrm{Spa}(A,A^+))(T/s) = \mathrm{Spa}(A,A^+) \cap (\mathrm{Spv}(A))(T/s)$$

is *only* being asserted when $T \cdot A$ is an open ideal. In general although $(\mathrm{Spv}(A))(T/s)$ is quasi-compact for any $s \in A$ and finite subset $T \subset A$ (Theorem 1.16), there is no reason that its intersection with $\mathrm{Spa}(A,A^+)$ should be quasi-compact with general finite subsets $T \subset A$.

Indeed, such quasi-compactness can fail, even for $T = \{s\}$. That is, the locus in $X = \mathrm{Spa}(A,A^+)$ where $v(s) \neq 0$ for a general $s \in A$ is generally not quasi-compact. For example, if A is Tate with a pseudo-uniformizer ϖ then the condition $v(s) \neq 0$ is equivalent to the condition $v(s) \geq v(\varpi^n)$ for some $n \geq 1$ (which forces $v(s) \neq 0$ since $v(\varpi)$ must be nonzero due to ϖ being a unit in A). Thus, $X(s/s)$ is the union of the (quasi-compact!) rational domains $X(\varpi^n/s)$ but it is typically not the union of finitely many of these (so $X(s/s)$ is then not quasi-compact); e.g., this occurs for A an affinoid algebra over a non-archimedean field and s a nonzero nonunit in A.

It cannot be overemphasized how important is the robustness of the "rational domain" concept inside X versus $X(T/s)$ as at the end of Theorem 2.14; this is used all the time without comment, and its proof is very hard (for the implication that rational domains in $X(T/s)$ are also rational in X, which comes down to showing that one can slightly modify the parameters describing a rational domain without affecting the actual subset defined inside $\mathrm{Spa}(A,A^+)$).

Recall that by definition a base for the topology of a valuation spectrum $Y = \mathrm{Spv}(R)$ for any ring R is given by (non-trivially quasi-compact) open subsets $Y(T/s)$ for $s \in R$ and finite subsets $T \subset R$. An important and highly non-trivial refinement for a Huber pair (A,A^+) and the corresponding *adic spectrum* $X = \mathrm{Spa}(A,A^+)$ is that a base for its topology is given by those subsets $X(T/s)$ for which the ideal $T \cdot A$ is *open* in the topology of the non-archimedean ring A. In other words:

Theorem 2.16. *For any pair (A,A^+) as in Theorem* 2.14*, the rational domains are a base for the topology of the spectral space* $\mathrm{Spa}(A,A^+)$.

This is not an easy result. Via (1), it is deduced from a purely algebraic analogue in [C3, 9.2.5(2)] giving a refined base for the topology of the subset $\mathrm{Spv}(R,J) \subset \mathrm{Spv}(R)$ (mentioned just below Theorem 2.16) for any commutative ring R and ideal $J \subset R$ whose radical coincides with the radical of a finitely generated ideal.

Remark 2.17. We have noted that the spectrality of $X = \mathrm{Spa}(A,A^+)$ for Huber pairs (A,A^+) and so also of its rational domains $X(T/s) = \mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+)$ ultimately rests on the spectrality of general valuation spectra $\mathrm{Spv}(R)$ for commutative rings R and of the subspace $\mathrm{Cont}(A) \subset \mathrm{Spv}(A)$ of continuous valuations for any Huber ring A. The spectrality of these basic building blocks of adic spaces (see Definition 3.12) is an essential difference from Berkovich spaces, whose affinoid building blocks are compact Hausdorff spaces (which are essentially never spectral). The overall proof of the spectrality of adic spectra is very long, much more so than quasi-compactness results for Zariski spectra of rings and for Berkovich spectra of classical affinoid algebras over non-archimedean fields (which is ultimately a short argument using Tychonoff's theorem). Hence, one may naturally wonder if it is worth the effort. Or more concretely: of what use is the spectral property?

There are two essential features of spectrality: the abundant supply of quasi-compact open subspaces, and the availability of specialization arguments (due to the property of being sober, which ensures that distinct points have distinct closures). The utility of such notions is very familiar from experience with schemes, including the usefulness of constructibility as a midway point towards openness and closedness results (since one can characterize when a locally constructible subset of a scheme is open or closed in terms of specialization relations). Moreover, in the development of étale cohomology one uses specialization techniques all over the place when setting up the theory.

Consequently, it is reasonable to expect (and indeed turns out to be the case) that the spectral property of $\mathrm{Spa}(A, A^+)$ and its rational subsets should be equally useful in the study of adic spaces. However, whereas one can work with schemes without being conscious of the notion of spectral space (because the Zariski topology of the spectrum of a ring is so closely related to calculations in the ring), for topological arguments with adic spectra $\mathrm{Spa}(A, A^+)$ (whose closed sets are *not* built from ideals in A, and whose "coordinate rings" for rational domains involve a completion operation) one needs the general theory of spectral spaces in order to get any non-trivial control over the topology. Once that tool is in hand, we get several benefits as follows.

Firstly, as we know from experience with spectra of rings, the ubiquity of quasi-compact open subsets is a powerful substitute for intuition with locally compact Hausdorff spaces. In particular, when aiming to bootstrap from calculations with stalks of the structure sheaf to global properties of a space, it is extremely useful that one has both a base of quasi-compact open subsets as well as a rather explicit such base (namely, the rational domains) in the affinoid setting. In effect, this substitutes for the finiteness property that Tate inserted manually into his "admissible topology", but is much more convenient because it involves an actual topological space (not a Grothendieck topology); this provides the ability to make pointwise arguments, which is only available in Tate's framework in very special circumstances (such as with coherent sheaves, which are too limited for developing an adequate theory of étale cohomology for non-archimedean geometry).

Furthermore, the property of being sober underlies a well-behaved theory of constructible subsets (Boolean combinations of quasi-compact open subsets) with which one can describe certain closures in terms of pointwise specialization relations. This is reminiscent of using stability under specialization to characterize closedness of a constructible subset of a noetherian scheme. One of Huber's primary motivations when setting up his theory of adic spaces (and also an essential tool in Scholze's work with perfectoid spaces) was to create a version of étale cohomology for rigid-analytic spaces. It is hopeless to create a theory of étale cohomology directly with Tate's rigid-analytic spaces, much as one can't make étale cohomology of classical algebraic varieties: one must use the associated schemes because specialization considerations pervade the proofs of all of the serious theorems in étale cohomology, and those arguments can't be conveniently expressed in the language of classical varieties (with only closed points). One should view the adic space associated to a classical rigid-analytic space (see Example 3.15) as akin to the scheme associated to a classical algebraic variety, and so the pervasive role of specialization arguments in work with étale cohomology for schemes is substituted by the good behavior of specialization relations in spectral spaces when developing étale cohomology for adic spaces.

(In Berkovich setting there are no specialization arguments because his spaces are locally compact. To overcome this in his development of étale cohomology for Berkovich spaces, he uses many arguments with curve fibrations and the nice topological features of locally compact Hausdorff spaces. There is however a price to pay: in addition to the local compactness of the underlying topological space, Berkovich needs to use a mild Grothendieck topology for considerations with coherent sheaves, somewhat reminiscent of Tate's admissible topology. This makes certain aspects of sheaf theory on Berkovich spaces somewhat subtle, and forces the notions of étale and smooth morphism in the Berkovich setting to be somewhat more restrictive than for adic spaces attached to rigid-analytic spaces. Of course, with experience one can get acclimated to these things just as one does with the admissible topology for rigid-analytic spaces.)

A companion to the notion of spectral space is:

Definition 2.18. A *spectral map* $f : X \to Y$ between spectral spaces: is a continuous map such that for every quasi-compact open subset $U \subset Y$ the open subset $f^{-1}(U)$ of X is quasi-compact.

It is obvious that for any map of commutative rings $A \to B$ the associated map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is spectral. However, if $f : (A, A^+) \to (B, B^+)$ is a map of Huber pairs (always required to be continuous on underlying topological rings) then although the induced map $\mathrm{Spa}(f) : X = \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) = Y$ is trivially continuous (since $\mathrm{Spv}(f)$ is trivially continuous and even spectral, as the preimage of $\mathrm{Spv}(A)(T/s)$ is $\mathrm{Spv}(B)(f(T)/f(s)))$ it is typically *not* spectral.

The issue is that when working with rational domains in Y, if $s \in A$ is an element and $T \subset A$ is a finite subset such that the ideal $T \cdot A$ is open in A then there is no reason that the ideal $f(T) \cdot B$ in B is open. Indeed, this fails for examples arising from the setting of "non-adic" maps of noetherian formal schemes (i.e., maps for which an ideal of definition does not pull back to an ideal of definition):

Example 2.19. Let A and B be noetherian rings equipped with topologies defined by respective ideals $I \subset A$ and $J \subset B$. Taking each ring to be its own ring of definition, we get Huber pairs (A, A) and (B, B). Suppose $f : A \to B$ is a ring homomorphism such that $f(I) \subset J$, so $f : (A, A) \to (B, B)$ is a (continuous!) map of Huber pairs.

For a finite set T of generators of I (so $T \cdot A = I$ is an open ideal of A), the ideal $f(T) \cdot B = IB$ usually does not contain a power of J and so is not open. Thus, for $s \in A$ the $\mathrm{Spa}(f)$-preimage of the rational domain $\mathrm{Spa}(A, A)(T/s)$ is the open subset $\mathrm{Spa}(B, B)(f(T)/f(s))$ that has no reason to be a rational domain and in fact is usually not even quasi-compact.

A situation where such quasi-compactness fails is $A = B = R[\![t]\!]$ as rings for a discrete valuation ring R, $I = 0$, $J = tB$, f is the identity map, $T = \{0\}$, and $s = t$.

There are related continuous maps of more serious interest that also fail to be spectral:

Example 2.20. For any Huber pair (A, A^+), consider the "support map" $X := \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ that assigns to each v its kernel $\mathfrak{p}_v = \ker(v) = \{a \in A \mid v(a) = 0\}$. This is continuous since the preimage of $\mathrm{Spec}(A_s)$ is the subset $\{v \in X \mid v(s) \neq 0\} = X(s/s)$ of X that is certainly open but is typically *not* quasi-compact, as we noted near the end of Remark 2.15.

For any Huber pair (A, A^+) and $x \in \mathrm{Spa}(A, A^+)$, the associated map $A \to k(x)$ to a complete valued fields carries A^+ into its valuation ring R_x; i.e., we have an associated map $\phi_x : A^+ \to R_x$. In the Tate setting, so $A^{00} \subset A^+$ since A^+ is open and integrally closed, we have $A = A^+[1/\varpi]$ for a pseudo-uniformizer ϖ and necessarily ϕ_x carries ϖ to a pseudo-uniformizer of R_x (as necessarily $0 < v(\varpi) < 1$, since $\varpi \in A^\times \cap A^{00}$ with $v \in \mathrm{Cont}(A)$ by definition of $\mathrm{Spa}(A, A^+)$). Thus, we can interpret Example 2.20 as associating to each x the image of the generic point η_x under

$$\mathrm{Spec}(\phi_x) : \mathrm{Spec}(R_x) \to \mathrm{Spec}(A^+)$$

(carrying η_x into the open subscheme $\mathrm{Spec}(A) = \{z \in \mathrm{Spec}(A^+) \mid z(\varpi) \neq 0\}$ since $|\varpi(x)| \neq 0$).

At the opposite extreme, we can consider the image $\mathrm{sp}(x)$ of the closed point under $\mathrm{Spec}(\phi_x)$. This does *not* land in the open subscheme $\mathrm{Spec}(A)$ since $\phi_x(\varpi)$ belongs to the maximal ideal of R_x due to the condition $|\varpi(x)| < 1$. In view of the Tate condition, the closed complement $\mathrm{Spec}(A^+) - \mathrm{Spec}(A)$ is precisely

$$\mathrm{Spec}(A^+/(\varpi)) = \mathrm{Spec}(A^+/A^{00}) = \mathrm{Spf}(A^+),$$

the underlying topological space of open prime ideals of A^+. Here is an interesting property of this construction:

Proposition 2.21. *For a Tate pair (A, A^+), the preceding "specialization map" of spectral spaces*

$$\mathrm{sp} : X = \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A/A^{00}) = \mathrm{Spf}(A^+)$$

is continuous and in fact spectral.

Proof. We just have to check that for any $a \in A^+$ representing an element $\overline{a} \in A^+/A^{00}$, the preimage $\mathrm{sp}^{-1}(D(\overline{a}))$ is open and quasi-compact. In fact, we claim this preimage is even a rational domain: it is $X(1/a)$. To see this, we just observe that for any $x \in X$, by definition $\mathrm{sp}(x)$ corresponds to the pullback of the maximal ideal of R_x under $\phi_x : A^+ \to R_x$, so $x \in \mathrm{sp}^{-1}(D(\overline{a}))$ if and only if the associated map from A^+/A^{00} into the residue field of R_x doesn't kill $\overline{a}$. It is equivalent to say that the image $\phi_x(a) \in R_x$ is a unit, or in other words $|a(x)| = 1$. But we know that always $|a(x)| \leq 1$ (since $A \to k(x)$ carries A^+ into the valuation ring R_x because $x \in \mathrm{Spa}(A, A^+)$), so the condition $|a(x)| = 1$ is the same as the condition $|1(x)| \leq |a(x)| \neq 0$, which says exactly that $x \in X(1/a)$. ■

Via the natural identification of sets $\mathrm{Cont}(A) = \mathrm{Cont}(\widehat{A})$, one also shows the following (see [C3, 11.5.1] for details):

Theorem 2.22. *The natural map* $\mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism respecting the notion of rational domain on each side.*

Remark 2.23. Typically the open subring $\widehat{A^+} \subset \widehat{A}$ is not integrally closed. Its integral closure is denoted as $\widehat{A}^+$ (abusing notation, since it depends on A and A^+), so we could also express the preceding theorem using $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ on the left.

We also have an evident homeomorphism $\mathrm{Spa}(A/\overline{\{0\}}, A^+/\overline{\{0\}}) \simeq \mathrm{Spa}(A, A^+)$ respecting the notion of rational domain on both sides, and $A/\overline{\{0\}}$ is a Hausdorff Huber ring. Using his brilliant skill with valuation theory, Huber proved the following result highlighting for the first time in the development of the general theory why one should limit attention to the case $A^+ \subset A^0$:

Theorem 2.24. *Assume A is Hausdorff and $A^+ \subset A^0$. Then* $\mathrm{Spa}(A, A^+)$ *is empty if and only if* $A = 0$.

The proof of this important theorem is given as [C3, 11.6.1]. The idea of the proof of the interesting implication "$\Rightarrow$" is to use that $A^+ \cdot A^{00} \subset A^{00}$ (as $A^+ \subset A^0$!) to deduce from emptiness of $\mathrm{Spa}(A, A^+)$ that $I^n = I^{n+1}$ for sufficiently large n. But then the Hausdorff condition on A forces the terminal power I^n for large n to vanish, so the topology on A is discrete and hence we may easily conclude.

Further mastery with valuation theory was used by Huber to prove:

Theorem 2.25 (adic Nullstellensatz)**.** *For any open and integrally closed subring $A^+ \subset A$, we have*

$$A^+ = \{a \in A \,|\, v(a) \leq 1 \text{ for all } v \in \mathrm{Spa}(A, A^+)\}.$$

The merit of allowing *non-complete* A is that it permits us to compute $\mathrm{Spa}(\widehat{A}, \widehat{A^+})$ by using automorphisms of A that might not pass to the completion. For example, if we want to describe the adic closed unit disc

$$\mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle) = \mathrm{Spa}(k[t], k^0[t]) \subset \mathrm{Cont}(k[t])$$

then we can applying automorphisms of $k[t]$ which may not even preserve $k^0[t]$; this really is useful when determining all points of that disc for algebraically closed k. Likewise, one builds some perfectoid algebras as completions of certain direct limits $A = \varinjlim A_i$, so being able to work with $\varinjlim A_i$ (rather than only a completion thereof) is convenient for some calculations.

Example 2.26. Let's now describe all points of $\mathbf{D}_k := \mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle)$ when k is algebraically closed. Our interest is to go beyond the classical points ("type 1"), so we focus on those v whose support is (0). These fall into 4 classes, three of rank 1 ("types 2, 3, 4") and a fourth class ("type 5") of rank 2. The justification of the following as an exhaustive list is given in [C3, 11.2–11.3].

Points of $\mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle)$ lie over $\mathrm{Spa}(k, k^0)$ that is a 1-point space (as k^0 is a rank-1 valuation ring). In other words, the rank-1 value group $k^\times/(k^0)^\times$ (which we may denote as $|k^\times|$ if we *non-canonically* choose an embedding of it into $\mathbf{R}_{>0}$) naturally lies inside the value group Γ_v for every $v \in \mathbf{D}_k$; if this rank-1 group exhausts Γ_v then we may say informally that v has value group equal to $v(k^\times)$

First consider non-classical v of rank 1, and let κ denote the residue field of k^0. The "type 2" class consists of v for which $\Gamma_v = v(k^\times)$ and $\kappa(v) \neq \kappa$ (i.e., same value group as the unique point in $\mathrm{Spa}(k, k^0)$, but bigger residue field). These v have the form

$$v_{c,r}(f) = \sup_{|t-c|\leq r\in|k^\times|} |f(t)|,$$

(note the requirement $r \in |k^\times|$) and $\kappa(v) \simeq \kappa(t)$.

On the other hand, the rank-1 cases with $\Gamma_v \neq v(k^\times)$ (and $\kappa(v) \neq \kappa$) are called "type 3" and are given by $v_{c,r}$ with $r \in \mathbf{R}_{>0} - |k^\times|$. In these cases, $\Gamma_v = \langle |k^\times|, r\rangle \subset \mathbf{R}_{>0}$.

The remaining rank-1 possibility, called "type 4", occurs if and only if k is not spherically complete, and is the situation with $k(t)_v/k$ an immediate extension (i.e., same residue field, same value group). These are given by $\inf_i \|\cdot\|_{D_i}$ for a descending family of closed discs D_i with $\cap_i D_i = \emptyset$. (In [C3, 11.3.4] there is given such a $\{D_i\}$ for $k = \mathbf{C}_p$.)

Finally, we come to type 5. These are rank-2 points arising from type-2 points via (continuous) specialization: the residue field $\kappa(t)$ at a type-2 point $v_{c,r}$ admits as its *non-trivial* valuations trivial on κ precisely the non-generic points $w \in \mathrm{RZ}(\kappa(t)/\kappa) = \mathbf{P}^1_\kappa$ (avoiding $w = \infty$ when $|c| = 1$). This gives rise to a rank-2 valuation on $k(t)$ by exactly the same process as in Example 1.6!

3. Sheaves, adic spaces and points

Due to Theorem 2.24, *we now assume always that* $A^+ \subset A^0$ for any Huber pair (A, A^+) under consideration; many results below would need to be reformulated if this implicit hypothesis were not imposed throughout.

3.1. **Structure sheaf.** Let (A, A^+) be a Huber pair (with $A^+ \subset A^0$ as always now), and define $X = \mathrm{Spa}(A, A^+)$. For $s \in A$ and a finite subset $T \subset A$ such that the ideal $T \cdot A$ is open in A, let $W = X(T/s)$. Consider the Huber pair

$$\mathscr{O}_A(W) = A\langle T/s\rangle, \quad \mathscr{O}_A^+(W) = (A^+[T/s]^\wedge)^\sim \subset A\langle T/s\rangle;$$

the latter ring is by definition the integral closure of the open completed subring $A^+[T/s]^\wedge \subset A\langle T/s\rangle$, and is denoted $A\langle T/s\rangle^+$ (in accordance with Remark 2.23). For instance, we want to define $\mathscr{O}_A(X)$ to be $\widehat{A}$ (not A in general!). Strictly speaking we should write $\mathscr{O}_{A,A^+}$, but that would be too cumbersome.

There immediately arises a question: does this pair depend only on the subset $W \subset X$ and not on the specific pair (s, T) used to build W? We will address this affirmatively soon via a universal mapping property solely in terms of $W \subset X$ without reference to s and T. Before we take up this issue, we give an example to illustrate how the affirmative answer would break down if completions were not part of the construction:

Example 3.2. Consider $X = \mathbf{D}_k := \mathrm{Spa}(k[t], k^0[t])$. Note that $X = X(1/(1 + \varpi t))$ for $\varpi \in k$ satisfying $0 < |\varpi| < 1$, but $k[t][1/(1 + \varpi t)] \neq k[t]$! The key issue here is that $1 + \varpi t$ is nowhere-vanishing on X but is not a unit in $k[t]$.

The preceding example highlights the importance of completeness in the following Key Lemma:

Lemma 3.3. *Let (B, B^+) be a Huber pair with B separated and complete for its topology; i.e., $B = \widehat{B}$. Any $b \in B$ non-vanishing on $\mathrm{Spa}(B, B^+)$ is a unit in B.*

The basic idea of the proof of the lemma is as follows. By completeness, if (B_0, I) is a couple of definition for B then $1 + I \subset B^\times$. Hence, the subset $B^\times \subset B$ is open. It follows that if an element of B is arbitrarily close to units then it must be a unit. It is therefore enough to prove that the closed ideal $J := \overline{bB} \subset B$ exhausts B. But B/J is Hausdorff since J is closed, and by the hypotheses on b we see that $\mathrm{Spa}(B/J, B^+/(B^+ \cap J))$ is *empty*. Since $B^+/(B^+ \cap J) \subset (B/J)^0$ (why?), it follows from Theorem 2.24 that $B/J = 0$, concluding the proof of the lemma.

Inspired by the homeomorphism (2) in Theorem 2.14, both Lemma 3.3 and the adic Nullstellensatz underlie the proof of a universal property for the pair $(A\langle T/s\rangle, A\langle T/s\rangle^+)$ solely in terms of the subset $W := X(T/s) \subset X$ without reference to s and T:

Proposition 3.4. *For any Huber pair (B, B^+) with B separated and complete, and any continuous map of Huber pairs $f : (A, A^+) \to (B, B^+)$, $\mathrm{Spa}(f)$ lands inside $W := X(T/s)$ if and only if there exists a (necessarily unique) continuous factorization of f as:*

$$(A, A^+) \to (A\langle T/s\rangle, A\langle T/s\rangle^+) \to (B, B^+).$$

This universal property provides canonical (and transitive) restriction maps $\mathscr{O}_A(W) \to \mathscr{O}_A(W')$ for rational subsets $W, W' \subset X$ satisfying $W' \subset W$, and likewise for $\mathscr{O}_A^+$. In this way $\mathscr{O}_A$ and $\mathscr{O}_A^+$ constitute presheaves on the basis of rational domains in X, and we assemble them to presheaves on X itself via the recipes

$$\mathscr{O}_A(U) = \varprojlim_{W \subset U} \mathscr{O}_A(W), \quad \mathscr{O}_A^+(U) = \varprojlim_{W \subset U} \mathscr{O}_A^+(W).$$

To avoid total confusion in practice, we also need:

Proposition 3.5. *For $W = X(T/s) \subset X$ and $W' \subset W$ a subset that is rational with respect to the Huber pair $(A\langle T/s\rangle, A\langle T/s\rangle^+)$, necessarily W' is rational as a subset of X (i.e., with respect to (A, A^+)).*

This non-trivial result [H, Lemma 1.5(ii)] allows us to work *locally* when studying properties of $\mathscr{O}_A$ and $\mathscr{O}_A^+$ (i.e., to rename $(A\langle T/s\rangle, A\langle T/s\rangle^+)$ as (A, A^+)). In particular, combining this with Lemma 3.3 and the finite generatedness of I for any couple of definition (A_0, I) one obtains:

Corollary 3.6. *The stalk $\mathscr{O}_{A,x}$ is a local ring for any $x \in X$.*

By design, for any $x \in X$ we have a canonically associated valuation $v_x : \mathscr{O}_{A,x} \twoheadrightarrow \kappa(x) \to \Gamma_x \cup \{0\}$ (also denoted $f \mapsto |f(x)|$) and the valuation ring $\kappa(x)^+ \subset \kappa(x)$ is identified with $\mathscr{O}_{A,x}^+/\mathfrak{m}_x$; more specifically, the diagram of rings

$$\begin{array}{ccc} \mathscr{O}_{A,x}^+ & \longrightarrow & \mathscr{O}_{A,x} \\ \downarrow & & \downarrow \\ \kappa(x)^+ & \longrightarrow & \kappa(x) \end{array} \tag{3}$$

is Cartesian. (See [C3, 14.3.2] for further details on this.)

These valuations on the local stalks are used in the following important application of the adic Nullstellensatz over rational domains in X:

Proposition 3.7. *For any open subset $U \subset X$,*

$$\mathscr{O}_A^+(U) = \{f \in \mathscr{O}_A(U) \,|\, v_x(f) \leq 1 \text{ for all } x \in U\}.$$

We conclude that if $\mathscr{O}_A$ is a sheaf then so is $\mathscr{O}_A^+$. But is $\mathscr{O}_A$ a sheaf? We say that (A, A^+) is *sheafy* if $\mathscr{O}_A$ is a sheaf on $X = \mathrm{Spa}(A, A^+)$. The property of sheafyness really concerns covers of rational domains by finitely many rational domains (without confusion, due to Proposition 3.5), and *there are cases where it fails.* Huber gave some useful sufficient conditions for an affirmative answer:

Theorem 3.8. *The presheaf $\mathscr{O}_A$ is a sheaf if either $\widehat{A}$ has a noetherian ring of definition or if $A\langle t_1, \ldots, t_n\rangle$ is noetherian for all $n > 0$.*

The conditions in this Theorem only involves hypotheses on A rather than also A^+ (though the space X rests on A^+), and is applicable in particular to those A arising from noetherian formal schemes or rigid-analytic spaces. The noetherian conditions essentially always fail to apply to the Huber pairs arising for perfectoid spaces, so another sufficient criterion is needed, at least when A is Tate. This is provided by the following main result in [BV] (and whose proof is sketched in [C3, 13.6]):

Theorem 3.9 (Buzzard–Verberkmoes)**.** *If A is Tate and "stably uniform" (i.e., $\mathscr{O}_A(W)^0$ is bounded in $\mathscr{O}_A(W)$ for all rational domains $W \subset \mathrm{Spa}(A, A^+)$) then $\mathscr{O}_A$ is a sheaf.*

In such cases, $\mathrm{H}^i(W, \mathscr{O}_X) = 0$ for all $i > 0$ and rational domains $W \subset X$.

The proof of this result adapts Tate's ideas from the rigid-analytic case, and [BV, Example 4.6] provides an A that is uniform (i.e., A^0 is bounded in A) yet for which $\mathscr{O}_A$ is not a sheaf! Thus, the "stably uniform" condition really cannot be relaxed too much. A somewhat easier task is to give a uniform Tate ring A that is not stably uniform (but for which the sheaf property is not determined): [BV, Example 4.5] provides such an A (simpler than [BV, Example 4.6]).

The importance of Theorem 3.9 for us will that perfectoid rings, which are complete, Tate, and uniform by definition, are stably uniform. More specifically, it will be a non-trivial theorem that if A is perfectoid then the completed coordinate ring $A\langle T/s\rangle = \mathscr{O}_A(W)$ for any rational domain $W \subset \mathrm{Spa}(A, A^+)$ is again perfectoid (and in particular: uniform).

3.10. **Adic spaces and points.** Let $\mathcal{V}$ denote the category of triples $(X, \mathscr{O}_X, \{v_x\}_{x\in X})$ for $(X, \mathscr{O}_X)$ a topologically locally ringed space and v_x a valuation on the residue field $\kappa(x)$ of the local ring $\mathscr{O}_{X,x}$ for each $x \in X$. Morphisms are defined in the evident manner. Proposition 3.7 motivates:

Definition 3.11. For each such triple $(X, \mathscr{O}_X, \{v_x\}_{x\in X})$ and open subset $U \subset X$, define

$$\mathscr{O}_X^+(U) = \{f \in \mathscr{O}_X(U) \,|\, v_x(f) \leq 1 \text{ for all } x \in U\}.$$

The notion of *morphism* in $\mathcal{V}$ (with continuity conditions on the map of structure sheaves) is given in the evident manner.

Definition 3.12. An *adic space* is an object in $\mathcal{V}$ locally isomorphic to

$$(\mathrm{Spa}(A, A^+), \mathscr{O}_A, \{v_x\}_{x\in X}) \tag{4}$$

for sheafy Huber pairs (A, A^+). (Recall that we always demand $A^+ \subset A^0$!) Triples as in (4) will generally be denoted more succinctly as $\mathrm{Spa}(A, A^+)$, an abuse of notation that is familiar from the way we denote affine schemes.

An adic space is called *Tate* if it is covered by open subspaces $\mathrm{Spa}(A, A^+)$ for sheafy Huber pairs (A, A^+) such that A is Tate.

Proposition 3.13. *For any sheafy Huber pairs (A, A^+) and (B, B^+) with B separated and complete,*
$$\mathrm{Hom}((A, A^+), (B, B^+)) \to \mathrm{Hom}_{\mathcal{V}}(\mathrm{Spa}(B, B^+), \mathrm{Spa}(A, A^+))$$
is bijective.

Using that valuation rings are maximal with respect to (local) domination in their fraction field, one establishes:

Proposition 3.14. *For adic spaces $(X, \mathscr{O}_X, \{v_x\})$ and $(X', \mathscr{O}_{X'}, \{v'_{x'}\})$, a map $f : (X, \mathscr{O}_X) \to (X', \mathscr{O}_{X'})$ of topologically locally ringed spaces respects the valuation data (i.e., underlies a uniquely determined map in $\mathcal{V}$) if and only if the map $\mathscr{O}_{X'} \to f_*(\mathscr{O}_X)$ carries $\mathscr{O}^+_{X'}$ into $f_*(\mathscr{O}^+_X)$.*

In [C3, 16.1] one finds details justifying the following important example that makes the connection to rigid-analytic geometry:

Example 3.15. There is a unique fully faithful functor r_k from the category of rigid-analytic spaces over k to the category $\mathcal{V}_k$ of adic spaces over $\mathrm{Spa}(k, k^0)$ subject to the conditions that $r_k(\mathrm{Sp}(A)) = \mathrm{Spa}(A, A^0)$, r_k carries open immersions to open immersions, and $r_k(U \cap V) = r_k(U) \cap r_k(V)$ for admissible open $U, V \subset X$. Moreover, a collection $\{U_i\}$ of admissible open subsets of a rigid-analytic space X over k is an admissible cover if and only if $\{r_k(U_i)\}$ is an open cover of $r_k(X)$!

Hence, the viewpoint of adic spaces with their underlying (locally spectral) topology explains in a very satisfying way what is really going on with admissible opens and admissible covers introduced by Tate. In effect, this Grothendieck topology is somewhat akin to exploring the topology of the real line entirely in the language of the totally disconnected archimedean topology $\mathbf{Q}$ without ever knowing about $\mathbf{R}$, making it all the more remarkable that Tate was able to get as far as he did within the classical MaxSpec framework.

In the context of the preceding example, for any point $\xi \in r_k(X)$ it is clear that if ξ is a classical point (equivalently, $\kappa(\xi)$ is k-finite) then $\mathscr{O}_{r_k(X),\xi}$ coincides with the local ring on X at the corresponding classical point. In particular, such stalks of $\mathscr{O}_{r_k(X)}$ are noetherian and henselian due to the classical theory. (The henselian condition essentially expresses the k-analytic inverse function theorem, which also underlies the henselian property for local rings on complex-analytic spaces.) But are the stalks of $\mathscr{O}_{r_k(X)}$ noetherian at *all* points of $r_k(X)$? And how about the henselian property?

The noetherian question is an important one that appears not to have arisen in the earlier literature, but fortunately the answer is affirmative though not elementary: an elegant proof due to Michael Temkin is given in [C3, 15.1] using the excellence of affinoid algebras. As for the henselian property, there is an affirmative answer in broad generality for Tate adic spaces, once we make a definition:

Definition 3.16. A commutative ring R is *henselian* along an ideal J if for every étale R-algebra S, any section of $R/J \to S/JS$ lifts to a section of $R \to S$.

For local R with maximal ideal J this definition recovers one of several (non-trivially!) equivalent properties that are used to define the henselian property for local rings in EGA. Beyond the local case, the analogous several equivalent characterizations are given in [SP, Tag 09XI]. As an application of these non-trivially equivalent conditions, if I is an ideal of R contained in J and R is henselian along J then it is also henselian along I [SP, Tag 0DYD] (this fact is intuitively plausible via the analogy with completeness for noetherian rings along ideals, but not evident from the initial definition of a henselian pair).

Theorem 3.17. *Let X be a Tate adic space. Choose $x \in X$ and let*

$$\mathfrak{m}_x = \ker(\mathscr{O}_{X,x} \to k(x)) = \ker(\mathscr{O}^+_{X,x} \to k(x)^+)$$

be the corresponding maximal ideal of $\mathscr{O}_{X,x}$.

(i) *The pair $(\mathscr{O}^+_{X,x}, \mathfrak{m}_x)$ is henselian.*
(ii) *The pair $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$ is henselian.*

Proof. First we prove (i). Pick an open affinoid U around x on which there exists a topologically nilpotent unit t (as is provided by the Tate property of X). Then from the definition of $\mathfrak{m}_x$ and the non-vanishing of $|t(x)|$ we see that $\mathfrak{m}_x = t\mathfrak{m}_x \subset t\mathscr{O}^+_{X,x}$, so (i) reduces to the henselian property for $(\mathscr{O}^+_{X,x}, t\mathscr{O}^+_{X,x})$. By standard limit arguments, a direct limit of henselian pairs is a henselian pair. Thus, for (i) it suffices to prove the henselian property for each pair $(\mathscr{O}^+_X(W), t\mathscr{O}^+_X(W))$ for rational domains $W \subset U$ containing x.

In other words, we are reduced to showing that if (A, A^+) is a Tate pair that is *complete* and $t \in A$ is a pseudo-uniformizer (so $t \in A^+$ because A^+ is open and integrally closed) then we claim that A^+ is t-adically henselian. By design of the completion of a Huber pair, A^+ is the integral closure of a certain t-adic completion. Since any finite subset of $A^+ \subset A^0$ can be put into a ring of definition, and all rings of definition have their topology equal to the t-adic topology (in the sense of commutative algebra) for which they are complete (due to the topological completeness of the Tate ring A), we are thereby reduced to the henselian property of a ring relative to an ideal for which the associated adic topology is separated and complete. But that in turn is elementary via successive approximation (use the characterization in terms of lifting certain residually monic factorizations of single-variable polynomials, as in the classic setting of complete local noetherian rings).

For the deduction of (ii) from (i) via artful use of a pseudo-uniformizer and some diagram-chasing we refer the reader to the proof given in [Bh, 7.5.5(6)], ■

Motivated by the case of schemes, we want to relate points of adic spaces to morphisms from objects of the form $\mathrm{Spa}(F, F^+)$ for suitable valued fields F. As motivation for the correct class of such (F, F^+) to consider, we first take a look at the distinction between the residue field $\kappa(x)$ and its v_x-completion for points x in an adic space X. Already for the Berkovich space $M(A)$ associated to a classical affinoid algebra, it is familiar that the residue field of the stalk of the structure sheaf at a non-classical point ξ does not have an easy algebraic description directly in terms of A, and that the *completion* of that residue field is a more useful invariant than the actual residue field.

Here is an important and non-obvious fact:

Theorem 3.18. *If X is a Tate adic then then each point of X admits a unique rank-1 generization.*

The proof of this result immediately reduces to the affinoid case $X = \mathrm{Spa}(A, A^+)$, which is [C3, 8.3.3, 9.1.5]. The idea is as follows. For any any continuous valuation $v : A \to \Gamma \cup \{0\}$ bounded by 1 on A^+, by continuity of v a pseudo-uniformizer of A carried to a *topologically nilpotent* nonzero element in the associated valuation topology on $\mathrm{Frac}(A/\ker(v))$. Thus, this topology coincides with that of a rank-1 valuation on the same field by Proposition 1.8. In more concrete term, the subring of power-bounded elements of this field relative to the v-topology is a rank-1 valuation ring. The real work in the proof of Theorem 3.18 is to show that this rank-1 valuation with the same kernel as v is really *continuous* on A. That in turn rests on the almost purely algebraic description of continuity of valuations in (1).

Since valuations admitting a rank-1 generization on the same fraction field are precisely those with a nonzero topologically nilpotent in the fraction field (Proposition 1.8), for any Tate adic space

X the completed residue field $k(x) := \widehat{\kappa(x)}$ at each point $x \in X$ is a non-archimedean field (even if v_x is a higher-rank valuation, or in other words the completion $k(x)^+ = \widehat{\kappa(x)^+}$ of the valuation ring for v_x is a proper subring of the rank-1 valuation ring $k(x)^0$ of power-bounded elements of $k(x)$). This brings us to:

Definition 3.19. An *affinoid field* is a pair (k, k^+) where k is a non-archimedean field and k^+ is an open valuation subring of k^0.

Exercise 3.20. Show that if (k, k^+) is an affinoid field then it is a Huber pair when k is given the topology for its non-archimedean absolute value. Such a pair is thereby sheafy (such as by Huber's criteria, among other reasons), so $\mathrm{Spa}(k, k^+)$ makes sense as an adic space and the given absolute value on k is the unique generic point. (When k^+ is a higher-rank valuation ring, such adic spaces have more than a single point!)

Proposition 3.21. *Let X be a Tate adic space. The map of sets*

$$|X| \to \{f : \mathrm{Spa}(k, k^+) \to X \mid \kappa(x) \textit{ is dense in } k\}$$

(with (k, k^+) a varying affinoid field) defined by $x \mapsto (\mathrm{Spa}(\widehat{\kappa(x)}, \widehat{\kappa(x)^+}) \to X)$ is bijective, with inverse assigning to any f the image of the closed point.

We emphasize that in contrast with schemes, in this description of the underlying via maps from "test objects" we are using test objects whose underlying space typically has more than one point (when k^+ is a higher-rank valuation ring). This makes the study of fibers of morphisms between adic spaces exhibit features that deviate somewhat from experience with fibers of maps of schemes.

4. A mapping property for certain proper birational maps

To finish our discussion of the basics of adic spaces for the purposes of this workshop, we want to discuss a certain factorization property for sheafy Tate pairs (A, A^+) (to be applied in the perfectoid setting, so we definitely cannot make any noetherian assumptions!).

Let $X = \mathrm{Spa}(A, A^+)$ as a topological space; this is equipped with two sheaves of rings $\mathscr{O}_X$ and $\mathscr{O}_X^+$ whose stalks at each point are local rings. Let X^+ denote the locally ringed space $(X, \mathscr{O}_X^+)$. If we invert ϖ on the structure sheaf of X^+ then we recover $(X, \mathscr{O}_X)$, but beware that the natural map of ringed spaces $(X, \mathscr{O}_X) \to (X, \mathscr{O}_X^+)$ is *not* a map of locally ringed spaces (as the inclusion of stalks $\mathscr{O}_{X,x}^+ \hookrightarrow \mathscr{O}_{X,x}$ is not local!).

We will find it convenient in several places below to use an observation of Tate from [EGA I, 1.8.1] (given near the end of [EGA II]):

Proposition 4.1 (Tate)**.** *For any ring R and locally ringed space (Z, O_Z), the map of sets*

$$\mathrm{Hom}((Z, O_Z), \mathrm{Spec}(R)) \to \mathrm{Hom}(R, O_Z(Z))$$

defined by "induced map on global functions" is a bijection.

Combing this with a review of the construction of fiber products of schemes via bootstrapping from the affine case then yields:

Corollary 4.2. *A fiber product of schemes is also a fiber product in the category of* all *locally ringed spaces.*

Applying Proposition 4.1 to the locally ringed space X^+ and the ring A^+, we see that there exists a unique map of locally ringed spaces

$$h : X^+ \to \mathrm{Spec}(A^+)$$

such that on global sections it induces the canonical composite map $A^+ \to \widehat{A}^+ =: \mathscr{O}_X^+(X)$.

Example 4.3. Consider a complete Tate pair (k, k^0) for a rank-1 valuation ring k^0. In this case X^+ is a 1-point space with stalk k^0, so $X^+ = \mathrm{Spf}(k^0)$ and h is the canonical map $\mathrm{Spf}(k^0) \to \mathrm{Spec}(k^0)$.

Our aim is to prove the following general fact:

Theorem 4.4. *For any proper map $f : Y \to \mathrm{Spec}(A^+)$ that restricts to an isomorphism over the open subscheme $\mathrm{Spec}(A) \subset \mathrm{Spec}(A^+)$, there is a unique factorization*

$$X^+ \xrightarrow{g} Y \xrightarrow{f} \mathrm{Spec}(A^+)$$

of h through f.

In the special case that $(A, A^+) = (k, k^0)$ for a non-archimedean field k, the assertion in the theorem amounts to the equality $Y(k^0) = Y(k)$ that is an instance of the valuative criterion for properness. The general case will be deduced from the valuative criterion for properness applied to general valuation rings.

Theorem 4.4 is motivated by Raynaud's approach to classical rigid-analytic geometry via "generic fibers" of certain formal schemes over the valuation ring, using the identification of $\mathrm{Spa}(B, B^0)^+$ for a classical affinoid algebra B with the inverse limit of all formal-scheme models of $\mathrm{Sp}(B)$ in the sense of Raynaud. This identification is too much of a digression to formulate here and is not needed for this workshop, so we will pass over it in silence.

Proof. We first prove uniqueness in a slightly wider generality, with X^+ replaced by any open subspace U. This step will not use the full strength of A^+-properness, but rather just the separatedness aspect. Letting $g_1, g_2 : U \rightrightarrows Y$ be two such factorizations, we want to prove $g_1 = g_2$. By Corollary 4.2, we can form a unique map $g = (g_1, g_2) : U \to Y \times Y$ over $\mathrm{Spec}(A^+)$ whose compositions with the projections are the g_i's. It suffices to show that g factors through the diagonal Δ_{Y/A^+}.

Factoring through a closed immersion of schemes is characterized in the category of all locally ringed spaces in terms of killing the quasi-coherent ideal under pullback. Hence, for uniqueness in the refined form we are aiming to establish, it suffices to check that the natural $\mathscr{O}_{Y\times Y}$-linear map $\theta : \mathscr{I}_\Delta \to g_*(\mathscr{O}_U)$ of sheaves on $Y \times Y$ vanishes.

Let ϖ be a pseudo-uniformizer of A (so $\varpi \in A^+$). The locally ringed space $(X, \mathscr{O}_X^+[1/\varpi])$ is the adic space $(X, \mathscr{O}_X) = \mathrm{Spa}(A, A^+)$ that is the preimage of the open subscheme $\mathrm{Spec}(A) = \mathrm{Spec}(A^+[1/\varpi])$ under h, and the g_i's coincide on this preimage since $Y_A = Y[1/\pi] \to \mathrm{Spec}(A)$ is an isomorphism by hypothesis. Hence, the $g^{-1}(\mathscr{O}_U)$-linear map $g^{-1}(\mathscr{I}_\Delta) \to \mathscr{O}_U$ corresponding to θ vanishes on stalks after inverting ϖ, so it vanishes since ϖ-multplication on $\mathscr{O}_X^+$ is injective (as ϖ is a global unit in $\mathscr{O}_X$). This implies $\theta = 0$ as desired, so uniqueness of the global factorization is proved with X^+ replaced by any open subspace.

For existence of a morphism $g : X^+ \to Y$ of locally ringed spaces over $\mathrm{Spec}(A+)$, we first prove a technical lemma:

Lemma 4.5. *It suffices to prove existence when Y is finitely presented over A^+.*

Proof. Since ϖ multiplies injectively on $\mathscr{O}_X^+$, if there is to exist a factorization $g : X^+ \to Y$ with a general Y as under consideration (perhaps not finitely presented over A^+) then it factors through the closed subscheme defined by the quasi-coherent ideal sheaf of ϖ-power torsion in $\mathscr{O}_Y$. That closed subscheme satisfies the same initial hypotheses as Y, so it is harmless to rename it as Y for the purposes of proving existence. In other words, we may assume ϖ-multiplication on $\mathscr{O}_Y$ is injective.

By [C1, Thm. 4.2], there is a closed immersion $Y \hookrightarrow Y'$ into a finitely presented and separated A^+-scheme Y'. Let $\{\mathscr{I}_\lambda\}$ be the directed system of finitely generated quasi-coherent ideal sheaves in $\mathscr{O}_{Y'}$ contained in $\mathscr{I}_Y$. The A^+-scheme $Y_\lambda \subset Y'$ defined by killing $\mathscr{I}_\lambda$ is finitely presented and separated over A^+, and $\{Y_\lambda\}$ is an inverse system with affine transition maps having inverse limit Y. Since $Y[1/\varpi]$ is trivially finitely presented over $A^+[1/\varpi] = A$ (as $Y[1/\varpi] \to \mathrm{Spec}(A)$ is even an isomorphism, by assumption), its ideal inside $\mathscr{O}_{Y'}$ is finitely generated. Thus, for large enough λ we have $Y_\lambda[1/\varpi] = Y$.

Now by replacing Y' with Y_{λ_0} for some large λ_0 we may arrange that $Y'[1/\varpi] = \mathrm{Spec}(A)$. The ideal of Y in $\mathscr{O}_{Y'}$ therefore vanishes after inverting ϖ, so Y is defined precisely by killing the ϖ-power torsion in $\mathscr{O}_{Y'}$ (since we have arranged that $\mathscr{O}_Y$ has vanishing ϖ-torsion). In particular, $\mathrm{Hom}_{A^+}(X^+, Y) = \mathrm{Hom}_{A^+}(X^+, Y')$, so if we can prove existence of the desired factorization using the possibly *non-proper* Y' in place of Y (note that we have arranged that at least $Y'[1/\varpi] \to \mathrm{Spec}(A)$ is an isomorphism!) then we get it for Y as desired.

Since Y' is separated and finitely presented over A^+, by standard direct limit techniques from [EGA, IV$_3$] we can write $Y' = Y'_0 \otimes_R A^+$ for a sufficiently large noetherian subring $R \subset A^+$ containing ϖ and a separated R-scheme Y'_0 of finite type such that $Y'_0[1/\varpi] = \mathrm{Spec}(R[1/\varpi])$. Since R is noetherian, by the Nagata compactification theorem [C1] there exists a schematically-dense open immersion $j : Y'_0 \hookrightarrow Z_0$ into a proper R-scheme. Inverting ϖ preserves the schematic density condition, but the open immersion $j[1/\varpi]$ is a section (since $Y'_0[1/\varpi] = \mathrm{Spec}(R[1/\varpi])$ by design!) and hence is also a closed immersion. Being schematically dense, $j[1/\varpi]$ must therefore be an isomorphism. That is, $Z_0[1/\varpi] = \mathrm{Spec}(R[1/\varpi])$. We conclude that $Z' := Z_0 \otimes_R A^+$ is a finitely presented and *proper* A^+-scheme in which Y' is an open subscheme such that $Z'[1/\varpi] = \mathrm{Spec}(A)$.

The composite quasi-compact immersion $Y \hookrightarrow Y' \hookrightarrow Z'$ over A^+ must be a closed immersion since each of Y and Z' is A^+-proper, so Y is the inverse limit of the collection of finitely presented closed subschemes Z_λ of Z' containing Y, and clearly $Z_\lambda[1/\varpi] = \mathrm{Spec}(A)$ for all λ (by squeezing from the same for Y and Z'). Hence, once the finitely presented case is settled we get unique factorizations $X^+ \to Z_\lambda$ for each λ and these must be compatible (due to uniqueness). Thus, we can pass to the inverse limit to get a map $X^+ \to Y$ that does the job. ■

Now we may and do assume the A^+-proper Y is *finitely presented*. Due to the stronger form of uniqueness that has been proved, to show existence it suffices to work locally on X^+. Let $x \in X$ be a point of the underlying topological space, so $k(x)$ admits ϖ as a pseudo-uniformizer. The natural map $A^+ \to k(x)$ lands inside $k(x)^+$ since $\mathscr{O}_X^+(X) = \widehat{A}^+$, so the valuation ring $k(x)^+$ is thereby an A^+-algebra and hence $k(x)^+[1/\varpi]$ is an algebra over $A^+[1/\varpi] = A$. The equality $Y[1/\varpi] = \mathrm{Spec}(A)$ thereby provides a map

$$\mathrm{Spec}(k(x)) \to \mathrm{Spec}(A) = Y[1/\varpi] \subset Y$$

over $\mathrm{Spec}(A^+)$, and the valuative criterion for properness using the (essentially arbitrary) valuation ring $k(x)^+$ uniquely extends this to a map of *locally ringed spaces*

$$\mathrm{Spec}(k(x)^+) \to Y$$

over $\mathrm{Spec}(A)$. This latter map carries the closed point to some point of Y that we'll call $g(x)$.

Pick an affine open $\mathrm{Spec}(B) \subset Y$ around $g(x)$, so B is a finitely presented A^+-algebra (as we have arranged that Y is finitely presented over A^+!) and we have a local map $B_{g(x)} \to k(x)^+$ over A^+, which in turn defines a map of A^+-algebras $B \to k(x)^+$. Inverting ϖ yields an A-algebra map

$$\varphi : B[1/\varpi] \to k(x)$$

corresponding via Proposition 4.1 to a unque map of locally ringed spaces $\mathrm{Spa}(k(x), k(x)^+) \to \mathrm{Spec}(B[1/\varpi])$ over $\mathrm{Spec}(A)$.

We claim that the open subscheme $\mathrm{Spec}(B[1/\varpi]) \subset Y[1/\varpi] = \mathrm{Spec}(A)$ contains the image of x under the map of locally ringed spaces $h_\eta : X \to \mathrm{Spec}(A)$ defined via Proposition 4.1 similarly to how we built h. Indeed, since x lies in the image of the map of locally ringed spaces $\mathrm{Spa}(k(x), k(x)^+) \to X$ it suffices (by Proposition 4.1) to check that the natural map $A \to k(x)$ factors (necessarily uniquely) through $A \to B[1/\varpi]$, but this holds because φ is an A-algebra map. Having just proved that $X \to \mathrm{Spec}(A)$ carries x into the open subscheme $\mathrm{Spec}(B[1/\varpi])$, pullback on functions gives a map $\widetilde{\varphi} : B[1/\varpi] \to \mathscr{O}_{X,x}$ of A-algebras lifting φ. By design φ carries B into $k(x)^+$, so by the Cartesian property of the diagram

$$\begin{tikzcd} \mathscr{O}^+_{X,x} \arrow[r] \arrow[d] & k(x)^+ \arrow[d] \\ \mathscr{O}_{X,x} \arrow[r] & k(x) \end{tikzcd}$$

analogous to (3) we can uniquely fill in the indicated map F in a commutative diagram of rings:

$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d] & \mathscr{O}^+_{X,x} \arrow[r] \arrow[d] & k(x)^+ \arrow[d] \\ B[1/\varpi] \arrow[r, "\widetilde{\varphi}"'] & \mathscr{O}_{X,x} \arrow[r] & k(x) \end{tikzcd}$$

The upshot of this discussion is that we have lifted the initial map $B \to k(x)^+$ (as built above from the design of B!) to a map $F : B \to \mathscr{O}^+_{X,x}$! Moreover, this is a map of A^+-algebras because it suffices to check that after composing with the *inclusion* $\mathscr{O}^+_{X,x} \hookrightarrow \mathscr{O}_{X,x} = \mathscr{O}^+_{X,x}[1/\varpi]$ and using that $\widetilde{\varphi}$ is a map of algebras over $A = A^+[1/\varpi]$.

Now it is time to exploit that B is finitely presented over A^+. Letting U vary through the collection of rational domains around x in X, we have

$$\mathscr{O}^+_{X,x} = \varinjlim_U \mathscr{O}^+_X(U)$$

as A^+-algebras. Thus, by the finite presentation we can find some U so that F lifts to an A^+-algebra map $B \to \mathscr{O}^+_X(U)$. Invoking again Proposition 4.1, this latter ring map arises from a unique map of locally ringed spaces

$$U^+ \to \mathrm{Spec}(B)$$

over $\mathrm{Spec}(A^+)$. Composing with the inclusion of $\mathrm{Spec}(B)$ into Y thereby solves the factorization problem on the open subspace $U^+ \subset X^+$ around the point x that was arbitrarily chosen. These factorizations agree on overlaps due to the refined form of uniqueness that we proved at the start, so they glue to provide the desired factorization $g : X^+ \to Y$. This concludes the proof of Theorem 4.4. ■

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