PERFECTOID SPACES I

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1. Disclaimer

This is the transcription of a lecture given by Aise Johan de Jong for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture introduces the theory of perfectoid spaces, building upon the previous lectures on perfectoid fields, perfectoid algebras, and adic spaces.

2. Definitions

Fix a prime p, and $K, |\cdot|$ a perfectoid field with its valuation, and write $\mathfrak{m} = K^{\circ\circ} \subseteq K^\circ$ for the maximal ideal contained in the subring of power bounded elements. Let $K^\flat$ denote its tilt, and correspondingly write $\mathfrak{m}^\flat \subseteq K^{\flat\circ\circ} \subseteq K^{\flat\circ}$. Fix a pseudo-uniformizer $t \in K^{\flat\circ}$ such that $|p| \le |t| < 1$, and set $\varpi = t^\sharp$ to be the corresponding pseudo-uniformizer in K°.

2.1. **Definition.** A perfectoid affinoid K-algebra (R, R^+) is a Tate affinoid (K, K°)-algebra, with R perfectoid. This just means that R is a perfectoid K-algebra, and $R^+ \subseteq R^\circ$ is an open and integrally closed subring of the power bounded elements in R.

2.2. **Remark.**

(1) In particular, we have

$$\mathfrak{m}R^\circ = R^{\circ\circ} \subseteq R^+ \subseteq R^\circ,$$

and from this one can check that the cokernel of the map $R^+ \hookrightarrow R^\circ$ is killed by $\mathfrak{m}$, so $R^+ \to R^\circ$ is an almost isomorphism.

(2) R^+ is specified by choosing $\overline{R^+} = R^+/\mathfrak{m}R^\circ \subseteq R^\circ/\mathfrak{m}R^\circ$. This essentially means that choosing an open integrally closed subring R^+ of R° is the same as choosing an open integrally closed subring of $R^\circ/\mathfrak{m}R^\circ$.

With this definition, we have the tilting correspondence for perfectoid algebras, which says that

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[1]Typeset by Ashwin Iyengar

2.3. **Proposition** (Tilting Correspondence). *The category of perfectoid affinoid K-algebras is equivalent to the category of perfectoid affinoid $K^\flat$-algebras. Moreover, under this equivalence an affinoid K-algebra (R, R^+) is sent to $(R^\flat, R^{\flat+})$, where $R^\flat$ is the usual tilt, and $R^{\flat+}$ is determined by taking the open integrally closed subring of $R^\flat$ corresponding to the image of R^+ under the map*

$$R^\circ \to R^\circ/\mathfrak{m}R^\circ \xrightarrow{\sim} R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ}.$$

In this case, we have

(1) $R^{\flat+} = R^{+\flat}$ and

(2) R^+/p is semiperfect.

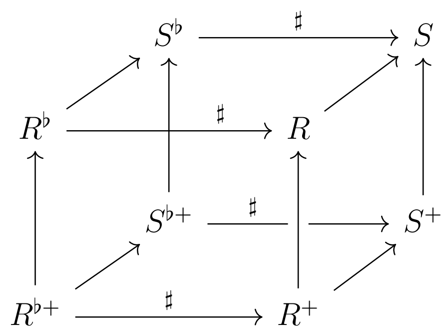
3. Tilting Rational Subsets

In this section we will give a proof of the fact that affinoid perfectoid spaces and their rational subsets "behave well after tilting".

If $\sharp$ denotes the untilt $R^\flat \to R$ or $R^{+\flat} \cong R^{\flat+} \to R^+$, then we have a commutative diagram

$$\begin{array}{ccc} R^\flat & \xrightarrow{\sharp} & R \\ \uparrow & & \uparrow \\ R^{\flat+} \cong R^{+\flat} & \xrightarrow{\sharp} & R^+ \end{array}$$

These are compatible with morphisms in the category of perfectoid affinoid K-algebras, in the sense that given a map $(R, R^+) \to (S, S^+)$, the functorially-obtained maps make the following diagram commute:



We come to the main theorem.

3.1. **Theorem.**

(1) For any valuation $x : R \to \Gamma \cup \{0\}$ in $\mathrm{Spa}(R, R+)$, the composition

$$R^\flat \xrightarrow{\sharp} R \xrightarrow{x} \Gamma \cup \{0\}$$

defines a point in $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$*. In fact, this process induces a homeomorphism*

$$(\cdot)^\flat : X \to X^\flat.$$

(2) *For* $U \subseteq X$ *a rational subset, the corresponding* $U^\flat \subseteq X^\flat$ *under the map* $(\cdot)^\flat$ *is also rational.*

(3) *The complete affinoid Tate algebra* $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ *is perfectoid, with tilt* $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$.

The proof of part (1) is easy-ish. The map $X \to X^\flat$ is well-defined because the untilt map $\sharp$ is a multiplicative map of monoids. One can show that the map is a bijection by showing that valuations on R or $R^\flat$ are parametrized by valuation rings on $R^\circ/\mathfrak{m}R^\circ \cong R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ}$. For continuity of the map, one can check that

$$((\cdot)^\flat)^{-1}\left(X^\flat\left(\frac{a_1, \dots, a_n}{b}\right)\right) = X^\flat\left(\frac{f_1, \dots, f_n}{g}\right),$$

where $f_i = a_i^\sharp$ and $g = b^\sharp$: this is basically immediate from the definitions. The harder part is showing that the inverse map is continuous. We can do this by showing that the image of a rational subset is in fact a rational subset.

First we state a lemma that fully describes Huber's presheaf taking values on rational open subsets pulled back from $X^\flat$.

3.2. **Lemma.** *If* $X = \mathrm{Spa}(R, R^+)$ *is any perfectoid affinoid* K*-algebra, then let* $U = X\left(\frac{f_1,\dots,f_n}{g}\right)$ *be a rational subset. Assume* $f_i = a_i^\sharp$, $g = b^\sharp$ *for* $0 < i < n$*, and* $a_n = t^N$ *and* $f_n = \varpi^N$ *for some* $N >> 0$*. Then let* $U^\flat = \left(\frac{a_1,\dots,a_n}{b}\right)$.

(1) *The* ϖ*-adic completion*

$$R^+\left\langle\left(\frac{f_i}{g}\right)^{1/p^\infty}\right\rangle = \widehat{R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]}$$

is a perfectoid $K^{\circ a} - algebra$.

(2) *The map*

$$R^+[x_i^{1/p^\infty}] \xrightarrow{\varphi} R^+\left[\left(\frac{f_i}{g}\right)^{1/p^\infty}\right]$$

taking $x_i^{1/p^m} \mapsto (f_i/g)^{1/p^m}$ *for all* m *is surjective, and the natural map*

$$(g^\epsilon x_i^\epsilon - f_i^\epsilon : \epsilon = 1/p^m \text{ for all } m) \to \ker\varphi$$

is an almost isomorphism.

(3) $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ *is a perfectoid affinoid* K*-algebra with tilt* $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$ *and*

$$R^+\left\langle\left(\frac{f_i}{g_i}\right)^{1/p^\infty}\right\rangle^a \cong_{K^{\circ a}} \mathscr{O}_X(U)^{\circ a}.$$

Proof. See [1], Lemma 9.2.5. □

Now we will state and prove the approximation lemma, which lets us nicely approximation elements of R by untilts (i.e. "perfect elements" in some sense). First we state the lemma for perfectoid polynomial rings.

3.3. **Lemma** (Approximation). *Fix $R = K\left\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\right\rangle$ and $f \in R^\circ$ an integral element, homogeneous of degree $d \in \mathbf{Z}_{(p)}$. Then for any $c \in \mathbf{Q}_{\geq 0}$ and any $\epsilon > 0$ there exists $g_{c,\epsilon} \in R^{\flat\circ}$ homogeneous of degree d such that*

$$\left|(f - g_{c,\epsilon}^\sharp)(x)\right| \leq |\varpi|^{1-\epsilon} \max(|f(x)|, |\varpi|^c) \text{ for all } x \in \operatorname{Spa}(R, R^\circ).$$

Proof. We do this using an inductive process. Fix $\epsilon > 0$: we will then drop it from $g_{c,\epsilon}$ and write g_c instead.

What we'll actually show is that for $c \in \mathbf{Q}_{\geq 0}$ you can find a $g_c \in R^{\flat\circ}$ homogeneous of degree d and some $\epsilon(c) > 0$ such that

$$\left|(f - g_c^\sharp)(x)\right| \leq |\varpi|^{1-\epsilon+\epsilon(c)} \max(|f(x)|, |\varpi|^c) \text{ for all } x \in \operatorname{Spa}(R, R^\circ),$$

which implies the result. We start with the case $c = 0$. In this case, the statement says that we wish to find $g_0 \in R^{\flat\circ}$ such that

$$\left|(f - g_0^\sharp)(x)\right| \leq |\varpi|^{1-\epsilon} \text{ for all } x \in \operatorname{Spa}(R, R^\circ).$$

To find such a g_0, note that $R^\circ/\varpi \cong R^{\flat\circ}/t$, so pick an element $g_0 \in R^{\flat\circ}$ which has the same image, under this quotient, as f. Then $f - g_0^\sharp = \pi h$ for some $h \in R^\circ$, so in fact

$$\left|(f - g_0^\sharp)(x)\right| = |(\varpi h)(x)| = |\varpi|\,|h(x)| \leq |\varpi| \text{ for all } x \in \operatorname{Spa}(R, R^\circ),$$

which concludes the base case.

Now suppose you've shown the result for c, so that we have an element $g_c \in R^{\flat\circ}$ and $\epsilon(c) > 0$ satisfying the result. To "induct", we want to be able to prove the theorem for $c' := c+a$, where $0 < a < \epsilon$. Since making $\epsilon(c)$ smaller weakens our result, we can assume that $\epsilon(c) < \epsilon - a$. We can define the rational open subset $U_c^\flat = X^\flat\left(\frac{g_c}{t^c}\right)$, whose pullback is the rational subset

$$U_c = X\left(\frac{g_c^\sharp}{\varpi^c}\right).$$

But if $x \in U_c$, then suppose $|f(x)| > |\varpi|^c$. Then

$$\begin{aligned} |f(x)| &\leq \max(\left|(f - g_c^\sharp)(x)\right|, \left|g_c^\sharp(x)\right|) \\ &\leq \max(|\varpi|^{1-\epsilon+\epsilon(c)} \max(|f(x)|, |\varpi|^c), |\varpi|^c) \\ &\leq \max(|\varpi|^{1-\epsilon+\epsilon(c)}|f(x)|, |\varpi|^c) \end{aligned}$$

But it can't be true that $|\varpi|^{1-\epsilon+\epsilon(c)}|f(x)| \leq |\varpi|^c$, so we must have the opposite, which by basic facts about ultrametric inequalities shows that $|f(x)| = |\varpi|^{1-\epsilon+\epsilon(c)}|f(x)|$, which can't be true (assume $\epsilon + \epsilon(c) \neq 1$ here). This, after swapping f with $g_c^\sharp$, shows that actually

$$U_c = X\left(\frac{f}{\varpi^c}\right).$$

Then note that

$$f - g_c^\sharp \in \varpi^{c+1-\epsilon+\epsilon(c)} \mathscr{O}_X^+(U_c)^\circ \cong_a \varpi^{c+1-\epsilon+\epsilon(c)} R^+ \left\langle \left(\frac{g_c^\sharp}{\varpi^c}\right)^{1/p^\infty} \right\rangle$$

and as $f - g_c^\sharp$ is homogeneous of degree d, it lies in

$$\bigoplus_{i \in \mathbf{Z}_{(p)}, 0 \leq i \leq 1} \varpi^{c+1-\epsilon+\epsilon(c)} \left(\frac{g_c^\sharp}{\varpi^c}\right)^i R^\circ_{\deg = d - di}.$$

So we can pick $r_i \in R^\circ$ homogeneous of degree $d - di$, where $r_i \to 0$ as $i \to \infty$ and all but finitely many $r_i = 0$, so that

$$f - g_c^\sharp = \sum_{i \in \mathbf{Z}_{(p)}, 0 \leq i \leq 1} \varpi^{c+1-\epsilon+\epsilon(c')} \left(\frac{g_c^\sharp}{\varpi^c}\right) r_i.$$

Notice here we put $\epsilon(c')$ instead of $\epsilon(c)$: this is an extra term from the fact that the almost isomorphism is with respect to the maximal ideal from the valuation topology on K. We need to look at the tilting situation, so pick elements $s_i \in R^{\flat +}$ homogeneous of degree $d - di$ with $s_i \to 0$ as $i \to \infty$, such that ϖ divides $r_i - s_i^\sharp$. Then we want to define

$$g_{c'} = g_c + \sum_{i \in \mathbf{Z}_{(p)}, 0 \leq i \leq 1} t^{c+1-\epsilon+\epsilon(c)} \left(\frac{g_c}{t^c}\right)^i s_i$$

Now it suffices to show that

$$|(f - g_{c'}^\sharp)(x)| \leq |\varpi|^{1-\epsilon+\epsilon(c')} \max(|f(x)|, |\varpi|^{c'}).$$

This is done in [2], Lemma 6.5.

□

We now summarize everything we want.

3.4. **Proposition.** *Let $(R, R^+)/(K, K^\circ)$ be as above. Then*

(1) For $f \in R$, $c \in \mathbf{Q}_{\geq 0}$, $\epsilon > 0$ there exists $g_{c,\epsilon} \in R^\flat$ such that for all $x \in X$ we get

$$|(f - g_{c,\epsilon}^\sharp)(x)| \leq |\varpi|^{1-\epsilon} \max(|f(x)|, |\varpi|^c).$$

(2) If $f, g \in R$ and $c \geq 0$, then there exists $a, b \in R^\flat$ such that

$$X\left(\frac{f, \varpi^c}{g}\right) = X\left(\frac{a^\sharp, \varpi^c}{b^\sharp}\right).$$

(3) For any $x \in X$, the completed residue field $\widehat{\kappa(x)}$ is a perfectoid field.

(4) $X \to X^\flat$ is a homeomorphism identifying rational subsets.

Proof. We prove (1). One can assume $R^+ = R_0$. By enlarging c a bit we assume that $f \in R^\circ$ and $c \in \mathbf{Z}_{\geq 0}$. Then we can write

$$f = g_0^\sharp + \varpi g_1^\sharp + \cdots + \varpi^c g_c^\sharp + \varpi^{c+1} f_{c+1}$$

for certain $g_i \in R^{\flat\circ}$ and $f_{c+1} \in R^\circ$. Then using the map

$$K\left\langle T_0^{1/p^\infty}, \ldots, T_c^{1/p^\infty}\right\rangle \to R$$

sending $T_i^{1/p^m} \mapsto (g_i^\sharp)^{1/p^m}$ for all m, we see that the image of $T_0 + \varpi T_1 + \cdots + \varpi^c T_c$ is exactly

$$f - \varpi^{c+1} f_{c+1}.$$

Now apply the approximation lemma to $\sum_i \varpi^i T_i$ to conclude the argument.

The rest of the argument can be found in [2].

□

References

[1] B. Bhatt, *Lecture notes for a class on perfectoid spaces.*

[2] P. Scholze, *Perfectoid Spaces*, Publ. Math. de l'IHÈS 116 (2012), no. 1, 245–313.