Perfectoid spaces II

MSRI, 3/13/18

K = perfectoid field, $\quad \mathfrak{m} \subset K^\circ \subset K$, $\quad \mathfrak{m} = \mathfrak{m}^2$

$K^\flat$ = tilt of K, $\quad \mathfrak{m}^\flat \subset K^{\flat\circ} \subset K^\flat$, $\quad \mathfrak{m}^\flat = \mathfrak{m}^{\flat 2}$

<u>Fix</u> a top. nilp. $t \in \mathfrak{m}^\flat \setminus \{0\}$, so $\begin{matrix} t^\# \in K^\circ \\ t \in K^{\flat\circ} \end{matrix}$ are pseudouniformizers. Normalize t so that $|p| < |t^\#| < 1$; then $K^\circ/t^\# \cong K^{\flat\circ}/t$.

<u>Def.</u> A perfectoid space $/K$ is an adic space $X/(K, K^\circ)$ that may be covered by opens $\mathrm{Spa}(A, A^+)$ with (A, A^+) an aff'd perf'd (K, K°)-alg.

<u>Will check</u>: any such (A, A^+) is sheafy.

The <u>tilt</u> of X is the perf'd space $X^\flat/K^\flat$ s.t.

$$X^\flat(R^\flat, R^{\flat+}) \cong X(R, R^+) \qquad \forall \text{ aff'd perf'd } (R, R^+)/K. \qquad (*)$$

<u>Rem.</u> 1. Any X has a unique tilt $X^\flat$: $|X| \cong |X^\flat|$ on top. spaces and one glues $X^\flat$ from $\mathrm{Spa}(A^\flat, A^{\flat+})$ (tilting equivalence for aff'd perf'ds $\Rightarrow (*)$).

2. The functor

$$\{\text{perf'd spaces}/K\} \xrightarrow{X \mapsto X^\flat} \{\text{perf'd spaces}/K^\flat\}$$

is an equivalence extending $\mathrm{Spa}(A, A^+) \mapsto \mathrm{Spa}(A^\flat, A^{\flat+})$.

3. <u>Warning</u>: it is unknown if a perfectoid X that is affinoid is Spa(aff'd perf'd).

<u>Sheafiness thm.</u> (Scholze) Fix an aff'd perf'd $(A, A^+)/K$, set $X := \mathrm{Spa}(A, A^+)$.

(a) $\mathcal{O}_X$ is a sheaf with $\Gamma(X, \mathcal{O}_X) \xleftarrow{\sim} A$.

$\Uparrow$ $\mathcal{O}_X^+$ ——"—— $\Gamma(X, \mathcal{O}_X^+) \xleftarrow{\sim} A^+$.

(b) (Tate acyclicity for aff'd perf'ds)

Each $H^n(X, \mathcal{O}_X)$ with $n > 0$ vanishes.

$\Uparrow$ Each $H^n(X, \mathcal{O}_X^+)$ with $n > 0$ is $\mathfrak{m}$-torsion.

<u>Suffices to prove</u>: $\forall$ finite covering $X = \bigcup U_i$ by rat'l subsets, each $H^n(C^\bullet)$ is $\mathfrak{m}$-torsion, where

$$C^\bullet := \left(\mathcal{O}_X^+(X) \to \bigoplus_i \mathcal{O}_X^+(U_i) \to \bigoplus_{i < i'} \mathcal{O}_X^+(U_i \cap U_{i'}) \to \dots \right)$$

with Čech differentials.

(<u>Then</u>: • Čech-to-derived ss $\Rightarrow$ (b),

• $C^\bullet[\frac{1}{t^\#}]$ exact $\forall X$ and $\forall \{U_i\}$ $\Rightarrow$ (a).)

Steps for this: 1. Use tilting to replace X/K by $X^\flat/K^\flat$.

2. Approximate $X^\flat/K^\flat$ by rigid spaces $Y/\mathbb{F}_p[[t]][\frac{1}{t}]$.

3. Conclude from Tate acyclicity for Y.

§1 Reduction to $X^\flat/K^\flat$

Prop. It suffices to prove that each $H^n(C^{\flat\bullet})$ is $\mathfrak{m}^\flat$-torsion, where

$$C^{\flat\bullet} := \left(\mathcal{O}^+_{X^\flat}(X^\flat) \longrightarrow \bigoplus_i \mathcal{O}^+_{X^\flat}(U_i^\flat) \longrightarrow \bigoplus_{i<i'} \mathcal{O}^+_{X^\flat}(U_i^\flat \cap U_{i'}^\flat) \to \dots\right).$$

Pf. By tilting [cf. prev. talk],

$$C^\bullet/t^\# \cong C^{\flat\bullet}/t \qquad \text{over} \qquad K^\circ/t^\# \cong K^{\flat\circ}/t.$$

So would get: each $H^n(C^\bullet/t^\#)$ is $\mathfrak{m}$-torsion

$\xRightarrow{\mathfrak{m}=\mathfrak{m}^2}$ each $H^n(C^\bullet/(t^\#)^N)$ is $\mathfrak{m}$-torsion

$\Longrightarrow$ each $m \in \mathfrak{m}$ kills the pro-systems $\{H^n(C^\bullet/(t^\#)^N)\}_{N>0}$

$\Longrightarrow$ each $H^n(C^\bullet) = H^n(\varprojlim_N (C^\bullet/(t^\#)^N))$ is $\mathfrak{m}$-torsion. $\square$

Rem. We are proving that each t^{1/p^N} kills each $H^n(C^{\flat\bullet})$, so wlog

$$K^\flat = (\mathbb{F}_p[t^{1/p^\infty}])^{\wedge_t}[\tfrac{1}{t}] \cong (\mathbb{F}_p[t]_{perf})^{\wedge_t}[\tfrac{1}{t}].$$

§2 Noetherian approximation

Prop. $\exists$ a filtered inductive system $\{B_j\}_{j\in J}$ of f.t., reduced, t-torsion free $\mathbb{F}_p[t]$-alg.'s with B_j int. closed in $B_j[\frac{1}{t}]$ s.t.

$$A^{\flat+} \cong \Big(\varinjlim_j \underbrace{((B_j)_{perf})^{\wedge_t}}_{\text{(integral) perfectoid } A_j}\Big)^{\wedge t} = \big(\varinjlim_j A_j\big)^{\wedge t}$$

Pf. Consider f.t. $\mathbb{F}_p[t]$-subalg.'s $B_j' \subset A^{\flat+}$, so B_j' is reduced, t-tor. free.

Set $B_j := \underbrace{(\text{integral closure of } B_j' \text{ in } B_j'[\tfrac{1}{t}])}_{\text{finite}/B_j'}$. b/c B_j' is excellent and reduced [EGA IV$_2$, 7.8.6 (ii)]

Then $A^{\flat+} \cong \varinjlim_j B_j \underset{\text{b/c } A^{\flat+} \text{ is perfect}}{\cong} \varinjlim((B_j)_{perf}) \underset{\text{b/c } A^{\flat+} \text{ is } t\text{-adically complete}}{\cong} \left(\varinjlim_j ((B_j)_{perf})^{\wedge_t}\right)^{\wedge_t}$ $\square$

Cor. Have $X^\flat = Spa(A^\flat, A^{\flat+}) \xrightarrow{\sim} \varprojlim_j \underbrace{Spa(A_j[\tfrac{1}{t}], A_j)}_{=: X_j}$,

compatibly with rat'l subsets:

each rat'l $V_j \subset X_j$ for some j
gives preimages $V_{j'} \subset X_{j'}$ for $j' \geq j$,
$V \subset X^\flat$

with $\mathcal{O}^+_{X^\flat}(V) \xleftarrow{\sim} \left(\varinjlim_j \mathcal{O}^+_{X_j}(V_j)\right)^{\wedge_t}$ (same univ. property)
and each V arises in this way for some (j, V_j) that depends on V. $\square$

<u>Rem.</u> In turn,

$$f_j : X_j = \mathrm{Spa}(A_j[\tfrac{1}{t}], A_j) \longrightarrow \mathrm{Spa}(B_j[\tfrac{1}{t}], B_j) =: Y_j$$

is a homeomorphism that identifies rat'l subsets. For rat'l $W \subset Y_j$,

$$\mathcal{O}^+_{X_j}(f_j^{-1}(W)) \xleftarrow{\sim} \left(\mathcal{O}^+_{Y_j}(W)_{\mathrm{perf}}\right)^{\wedge_t} \quad \text{(same univ. property).}$$

Moreover, Y_j is a classical rigid space / $\mathbb{F}_p[[t]][\frac{1}{t}]$ $(\Rightarrow \mathcal{O}^+_{Y_j} = \mathcal{O}^\circ_{Y_j})$!

<u>Upshot</u>: $C^{\flat\bullet} \cong \left(\varinjlim_j \left((C_j^\bullet)_{\mathrm{perf}}\right)^{\wedge_t}\right)^{\wedge_t}$

with each $C_j^\bullet$ a Čech complex of the form

$$C_j^\bullet \cong \left(\mathcal{O}_Y(Y)^\circ \xrightarrow{d^{-1}} \bigoplus_i \mathcal{O}_Y(V_i)^\circ \xrightarrow{d^0} \bigoplus_{i<i'} \mathcal{O}_Y(V_i \cap V_{i'})^\circ \xrightarrow{d^1} \dots\right)$$

for some aff'd rigid space $\underbrace{Y/\mathbb{F}_p[[t]][\tfrac{1}{t}] \text{ and a finite rat'l cover } Y = \bigcup_i V_i}_{\text{depend on } j}$.

§3 Conclusion of the argument

<u>Prop.</u> Each $H^n(C_j^\bullet)$ is killed by some t^m.

<u>Pf.</u> Tate acyclicity for aff'd rigid space Y $\Rightarrow$ $C_j^\bullet[\frac{1}{t}]$ is exact.
then Banach open mapping thm. $\Rightarrow$ $C_j^{n-1}[\frac{1}{t}] \xrightarrow{d^{n-1}[\frac{1}{t}]} \mathrm{Ker}(d^n)[\frac{1}{t}]$ is open.
$\Rightarrow t^m \cdot \mathrm{Ker}(d^n) \subset \mathrm{Im}(d^{n-1})$ for some m. $\square$

<u>Cor.</u> Each t^{1/p^N} kills every $H^n(C^{\flat\bullet})$. ($\Rightarrow$ Strafiness Thm.!)

<u>Pf.</u> Prop. $\Rightarrow$ t^{1/p^N} kills already each $H^n((C_j^\bullet)_{\mathrm{perf}}) \cong \varinjlim_\varphi (H^n(C_j^\bullet))$. $\square$

<u>Rem.</u> For a fixed Y, the exponent m in the Prop. may be unbdd:

<u>Eg.</u> (Bhatt) Set $A := \mathbb{F}_p[t, u, v]/(u^3 - v^2)$, so A is int. closed in $A[\frac{1}{t}]$.

$$Y := \mathrm{Spa}(A[\tfrac{1}{t}], A) \ni y_0 := \{u = v = 0\} = \{u = 0\} = \{v = 0\}.$$

Get $\frac{v}{u} \in H^0(Y - y_0, \mathcal{O}^+_Y) \subset H^0(Y - y_0, \mathcal{O}_Y)$.

For $U_n := \{ y \in Y \mid |y(u)| \leq |y(t)|^{2n} \}$, have $\frac{v}{u} \in t^n H^0(U_n \setminus \{y_0\}, \mathcal{O}_Y^+)$,
so $\frac{v}{u}$ extends by 0 over y_0 to $f_n \in H^0(Y, \mathcal{O}_Y^+/t^n)$.

$$0 \to \underbrace{\varprojlim_n \left(H^0(Y, \mathcal{O}_Y^+)/t^n \right)}_{\widehat{A}^t} \to \varprojlim_n \left(H^0(Y, \mathcal{O}_Y^+/t^n) \right) \to \underbrace{\varprojlim_n \left(H^1(Y, \mathcal{O}_Y^+)[t^n] \right)}_{\neq 0} \to 0$$

$$\widehat{A}^t \not\ni \{f_n\} \in \varprojlim_n \left(H^0(Y, \mathcal{O}_Y^+/t^n) \right) \qquad \underline{\text{Get}}:\ \neq 0$$

b/c $\frac{v}{u} \notin \hat{A}$ (one checks in $\widehat{A[\frac{1}{u}]}^t$)

$\Rightarrow H^1(Y, \mathcal{O}_Y^+)$ is not killed by any t^m.

§4 Dropping K

Often it is useful to work w/o a fixed perfectoid field K.

Eg. $R := \mathbb{Z}_p[[x, y]]/(p - x^2 - y^2)$ has a flat cover $R_\infty := \left(\mathbb{Z}_p[[x^{1/p^\infty}, y^{1/p^\infty}]]/(p - x^2 - y^2) \right)^{\wedge_p}$ that is "perfectoid", but $R_\infty[\frac{1}{p}]$ is not over a perfectoid field.

Def. A ring R is (integral) perfectoid if

- R is p-adically complete;
- $\exists\, \pi \in R$ with $\pi^p = pu$ for $u \in R^\times$ and $R/\pi \xrightarrow[x \mapsto x^p]{\sim} R/p$ (injectivity is automatic if R is int. closed in $R[\frac{1}{p}]$); and
- (not needed if $R[p] = 0$) the kernel of $\theta : W(R^\flat) \xrightarrow[{[x] \mapsto x^\sharp}]{} R$ is principal.

Rem. 1. Most results continue to hold w/o K: cf. Fontaine / Kedlaya–Liu (perfectoid Banach algebras $/\mathbb{Q}_p$ or $/\mathbb{F}_p$), Gabber–Ramero (perfectoid rings $/\mathbb{Z}_p$ in great generality), Bhatt–Morrow–Scholze (perfectoid rings $/\mathbb{Z}_p$).

2. The extent to which the tilting functor is an equivalence is governed by the Fargues–Fontaine curve.