COMPLEMENTS TO ALMOST PURITY

GERD FALTINGS

1. Disclaimer

This is the transcription of a lecture given by Gerd Faltings for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture gives some historical remarks on the almost purity theorem.

2. Historical Notes

Tate tried to compute the Galois cohomology $H^i(G, \mathbf{C}_p(n))$, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}_p}$, defined using continuous cochains. Given a coefficient module $M = \varprojlim M/p^n M$, then the continuous Galois cohomology is the derived direct image

$$H^*(G, M) = \mathbf{R} \varprojlim H^*(G, M/p^n M).$$

The idea was to relate this to Galois cohomology which you could compute, namely the first extension in the tower

$$\mathbf{Q}_p \subseteq \widehat{\mathbf{Q}_p(\mu_{p^\infty})} \subseteq \mathbf{C}_p,$$

In fact, the Galois cohomology of the first extension is just $\mathbf{Z}_p^\times$. Then the fact that the second extension doesn't change the Galois cohomology is, in modern terminology, a reflection of the fact that the second extension is "almost étale".

Now say we have a ring R with an action of a finite group G, and consider M a G-R-module, i.e. an R-module with a semilinear action of G (i.e. $g \cdot (rm) = (g \cdot r)(g \cdot m)$). Then $H^i(G, M)$ can be computed by a resolution of acyclic (G, R)'s. This is done by noting that there is an injective map $M \hookrightarrow \operatorname{Hom}(G, M)$, which then form an acyclic module for group cohomology.

You also know that $M^G \supseteq \operatorname{tr} M$, where $\operatorname{tr} = \sum_{g \in G} g$. Then if $x = \sum_g gy$ for $y \in R$, then x annihilates $H^i(G, M)$ for $i > 0$.

If V is a discrete valuation ring with fraction field K, and L/K is a finite Galois extension, then look at the normalization (integral closure) of V in L, called W. This is a semilocal ring, since the prime ideal in V can split. Say $G = \operatorname{Gal}(L/K)$ and fix a prime $\mathcal{P} \subseteq W$ over the prime $\mathfrak{p}$ in V. Then we have

$$I_P \subseteq D_P \subseteq G,$$

Date: March 14, 2018.

[1]Typeset by Ashwin Iyengar

which are the inertia and decomposition group in G. Assume V is complete, with algebraically closed residue field. If π is a uniformizer in V, then let Π be a uniformizer in W. Let $f(\Pi) = 0$ be the minimal equation over K, (which is an Eisenstein polynomial). Then it's known that

$$\Omega_{W/V} \cong W d\Pi/(f'(\Pi)),$$

where $d = [L : K]$, and that $\mathcal{D}_W$, the different, is $f'(\Pi)^{-1}W$. We normalize the length such that $l(W/pW) = 1$. Now we have $\delta_{W/V} = l(W/f'(\Pi)W)$.

This implies that if we have $V \subseteq W \subseteq W'$ we get

$$0 \to \Omega_{W/V} \otimes W' \to \Omega_{W'/V} \to \Omega_{W'/W} \to 0.$$

Now Tate's argument goes as follows.

Proposition 2.1. *Say you have $V \subseteq W$ fixed of degree d and a sequence $V_n \supseteq V$ with $\Omega_{V_n/V}$ "big" in the sense that $\delta_{V_n/V} \to \infty$. Then if W_n is the norm of $W \otimes_V V_n$, the claim is that $\delta_{W_n/V_n} \to 0$.*

Proof. Here is an elementary proof. If $V' \supseteq V$ and $\delta_{V'/V} \geq \delta_{W/V}$, then

$$\Omega_{W/V} \to \Omega_{W'/V'}$$

is 0, where $W' = N(W \otimes_V V')$. This is because you have two maps,

$$\Omega_{W/V} \otimes W' \to \Omega_{W'/V} \text{ and } \Omega_{V'/V} \otimes W' \to \Omega_{W'/V}$$

But these are all cyclic modules, so the submodules are ordered by length. But since $\delta_{V'/V} \geq \delta_{W/V}$ we must have that the image of $\Omega_{W/V} \otimes W'$ under the map to $\Omega_{W'/V}$ is contained in the image of $\Omega_{V'/V} \otimes W'$. Thus the map

$$\Omega_{W/V} \otimes W' \to \Omega_{W'/V}/(\Omega_{V'/V} \otimes W')$$

is 0, but the target is isomorphic to $\Omega_{W'/V'}$ by the above exact sequence.

Now look at $W \otimes_V W' \to W' \otimes_{V'} W'$. Let $I = \ker(W \otimes_V W' \to W')$, which is the base extension by W' of $W \otimes_V V' \xrightarrow{N} W'$ and J be the kernel of $W' \otimes_V W' \to W'$. Then we get a diagram

$$\begin{tikzcd}
W \otimes_V W' \arrow[r] & W' \otimes_{V'} W' \\
I \arrow[u, hook] \arrow[r] & J \arrow[u, hook] \\
I^2 \arrow[u, hook] \arrow[r] & J^2 \arrow[u, hook]
\end{tikzcd}$$

and the induced map $I/I^2 \to J/J^2$ is 0, so this descends to $I \to J^2$. This implies that the cokernel of $W \otimes_V W' \to W' \otimes_{V'} W'$ surjects onto J/J^2.

Now the length $\ell(\text{coker}) \geq \delta_{W'/V'}$. So $\delta_{W'/V'} = \delta_{W/V} - \frac{2}{d}\ell(\text{coker})$. So

$$\ell(\text{coker}) = \frac{d}{2}(\delta_{W/V} - \delta_{W'/V'}),$$

i.e.

$$\delta_{W'/V'} \leq \frac{d}{d+2}\delta_{W/V},$$

and now repeat this infinitely often. □

Now recall

Proposition 2.2 (Purity). *If R is a regular local ring, and $R \subseteq S$ is a finite normal extension, étale in codimension 1, then $R \subseteq S$ is an étale extension.*

This is proven in SGA2 by induction on $\dim R$. If $\dim R = 2$ then S is Cohen-Macaulay, thus projective over R. Then the discriminant of the trace form of S/R. But S/R is unramified at all height one primes, so the discriminant of $\mathrm{tr}_{S/R}$ is invertible in all height one primes of R, and thus by Krull's principal ideal theorem, it's invertible, and thus S/R is étale.

For $\dim R > 2$, then take a parameter $t \in R$ and consider R/tR and take the integral closure of S/tS: by induction this is étale over R/tR. By a theorem of Grothendieck, you can lift this to an étale cover S'/R, and we can get a map $S \hookrightarrow S'$, and the cokernel has dimension ≤ 1. But it can't be that the dimension is 1, so it must be 0. A full proof can be found in SGA2.

For almost purity, the same proof basically goes through for almost étale extensions in dimension 2. In higher dimensions, the idea is to look at hyperplane sections, and want to lift almost étale extensions. There is another argument that works for the classical purity theorem and then extends to the almost case almost immediately, by propagating a small error throughout.

The strategy is to take $\overline{S}$ over R/tR and lift to $\overline{S}$ as a projective module by taking the corresponding idempotent and lifting it over a nilpotent. But then you need to check that the multiplication $S \otimes S \xrightarrow{m} S$ lifts properly, i.e. is associative. But $m(a, m(b, c)) - m(m(a, b), c)$ defines a cocycle for Hochschild cohomology, which is trivial for étale extensions.

The point is that in the almost case, we just add in an error term. For example, we lift an almost idempotent, i.e. $\overline{e}^2 = p^\epsilon \overline{e}$, which lifts to $e^2 = p^{3\epsilon} e$, and for the Hochschild cohomology we use a similar technique, using a changing ϵ. At the end, we get a multiplication map $S_\epsilon \otimes S_\epsilon \to S_\epsilon$ lifting $p^\epsilon m$, and we get a multiplication on $\bigcup p^\epsilon S_\epsilon$, which is almost a lift of $\overline{S}$, whence the name almost.

Lastly, we discuss a proof of almost purity using Frobenius. The idea is essentially the following: If you have $R \subseteq S$ finite étale outside $\mathfrak{m}$, then the local cohomology is calculated as

$$H^i_{\mathfrak{m}}(S) = H^{i-1}(\operatorname{Spec} R \setminus \{\mathfrak{m}\}, \mathscr{O}_{\operatorname{Spec} S}).$$

In characteristic p, Frobenius induces an "isomorphism":

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\text{Frob}"] \arrow[dr] & \widetilde{S} \\
& \widetilde{S} \otimes_{R,\text{Frob}} R, \arrow[u]
\end{tikzcd}$$

where the right vertical map is an isomorphism away from the maximal ideal. Then

$$H^i_{\mathfrak{m}}(S) \cong H^i(S) \otimes_{R,\text{Frob}} R,$$

but this multiplies the length by p^{dim}. Somehow from this, you can deduce the result.

Apologies for the brevity and imprecision, but I found the argument hard to follow during later parts of the lecture.