ADJOINING p-POWER ROOTS OF UNITY

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1. Disclaimer

This is the transcription of a lecture given by Judith Ludwig for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture describes a step in the proof of Bhatt's the direct summand conjecture in [1], which involving adjoining pth roots of an element in a perfectoid algebra by constructing an almost faithfully flat extension.

2. Introduction

Let K be a perfectoid field, with $K^\circ \subseteq K$ as usual. Pick a pseudo-uniformizer $t \in K^\circ$ satisfying $|p| \leq |t| < 1$, which is the untilt of a pseudo-uniformizer $t_0 \in K^{\flat\circ}$ in the tilt. In particular, we have $t^{1/p^n} \in K^\circ$. In this talk, all occurrences of "almost" are with respect to $(t^{1/p^\infty}) \subseteq K^\circ$.

Our goal is the following: given an integral perfectoid K°-algebra A (i.e. $A[1/t]$ is a perfectoid K-algebra), and given any $g \in A$ we want to construct an extension $A \to A_\infty$ of integral perfectoid K°-algebras such that

(1) g admits arbitrary p-power roots in A_∞.

(2) The extension is almost faithfully flat mod t.

We will construct this extension explicitly using perfectoid geometry. "If the fog lifts... hopefully we will see the bridge....."

3. Zariski Closed Subsets

Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathrm{Spa}(K, K^\circ)$.

Definition 3.1. Let $|X|$ denote the underlying topological space of X. A subset $Z \subseteq |X|$ is "Zariski closed" if there exists an ideal $I \subseteq R$ such that

$$Z = \{x \in X : |f(x)| = 0 \text{ for all } f \in I\}.$$

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[1]Typeset by Ashwin Iyengar

Lemma 3.2. *Assume $Z \subseteq |X|$ is Zariski closed. Then there exists a universal perfectoid space $\mathcal{Z}$ over $\mathrm{Spa}(K, K^\circ)$ with a map $\mathcal{Z} \to X$ for which $|\mathcal{Z}| \to |X|$ factors through $Z \to X$. In fact, $\mathcal{Z}$ is affinoid perfectoid, equal to $\mathrm{Spa}(R_Z, R_Z^+)$, and the map $R \to R_Z$ has dense image, and $|\mathcal{Z}| \to Z$ is a homeomorphism.*

Proof. We can write $Z \subseteq |X|$ as a limit of open subsets.

$$Z = \bigcap_{Z \subseteq U \subseteq X, U \text{rational}} |U|.$$

Concretely, given $f_1, \ldots, f_n \in I$, let

$$U_{f_1,\ldots,f_k} = \{x \in X : |f_i(x)| \leq 1, i = 1, \ldots, k\}.$$

Then

$$Z = \bigcap_{\{f_1,\ldots,f_k\} \subseteq I} U_{f_1,\ldots,f_k}.$$

The containment $\subseteq$ is obvious. For the reverse containment, say you have $x \in |X| \setminus Z$. Then there exists $f \in I$ such that that $|f(x)| \neq 0$. By multiplying by a large enough power of t^{-1}, you will have

$$|t^{-\ell} f(x)| > 1,$$

so $x \notin U_{t^{-\ell} f}$. Note any $U_{f_1,\ldots,f_k} = \mathrm{Spa}(R_{f_1,\ldots,f_k}, R^+_{f_1,\ldots,f_k})$. Set

$$(R_Z, R_Z^+) = \varinjlim_{\{f_1,\ldots,f_k\} \subseteq I} (R_{f_1,\ldots,f_k}, R^+_{f_1,\ldots,f_k}),$$

i.e. where R_Z^+ is the t-adic completion of $\varinjlim R^+_{f_1,\ldots,f_k}$ and $R_Z = R_Z^+[t^{-1}]$. One can show that $\mathrm{Spa}(R_Z, R_Z^+)$ is universal. $\square$

Remark 3.3. For schemes, the corresponding statement of universality is wrong, due to the appearance of nilpotents. One can ask, how does R_Z compare to R/I? In fact, if we let $(R/I)^+$ denote the integral closure of R^+ in R/I, then $R_Z^+ \cong \widehat{(R/I)^+}$, where the hat denotes t-adic completion.

Remark 3.4 (Warning)**.** The map $R \to R_Z$ is not necessarily surjective in characteristic 0. For example, if $K\left\langle T^{1/p^\infty}\right\rangle$ and $\mathbf{Q}_{\mathrm{cyc}} \subseteq K$, then take $I = (T - 1)$. If we let

$$R_Z = C^0(\mathbf{Z}_p, K)$$

and let $R \to R_Z$ be polynomial evaluation, then $R \to R_Z$ is not surjective. However, $R \to R_Z$ is always surjective in characteristic p.

4. Constructing A_∞

Let A be an integral perfectoid K°-algebra, and take $X = \mathrm{Spa}(A[1/t], A)$. Then define

$$Y = \mathrm{Spa}(A\left\langle Y^{1/p^\infty}\right\rangle[1/t], A\left\langle Y^{1/p^\infty}\right\rangle).$$

One may think of this as the "perfectoid unit disk over X".

Let $g \in A$, and take $Z = V((T-g)) = \{y \in Y : |(T-g)(y)| = 0\} \subseteq Y$, defined by the ideal $(T-g) \subseteq A\left\langle T^{1/p^\infty}\right\rangle[1/t]$.

Furthermore, define $A_\infty = (A\left\langle T^{1/p^\infty}\right\rangle[1/t])^+_Z$ as the integral ring of functions on Z.

More explicitly, consider

$$Y_\ell = Y\left(\frac{T-g}{t^c}\right) = \{y \in Y : |(T-g)(y)| \leq |t^\ell|\} \subseteq Y.$$

for varying $\ell \in \mathbf{N}$. Then one can show that

$$A_\infty = \widehat{\varinjlim B_\ell},$$

for t-adic completion, where $B_\ell = \mathscr{O}_Y^+\left(Y\left(\frac{T-g}{t^\ell}\right)\right)$.

Now we have to verify the desired properties. First, we want to know that g admits p-power roots in A_∞. To see this, note that $T-g$ is divisible by t^ℓ in each B_ℓ, so $T-g$ is divisible by ℓ for all ℓ in A_∞, which can only happen when $T = g$ (topologically, in the sense of this limit). Thus, since T has p-power roots by construction, g does as well.

The second property is a theorem due to Yves André.

Theorem 4.1. *For each $\ell \geq 1$, the map $A \to B_\ell$ is almost faithfully flat mod t. Therefore,*

$$A \to A_\infty$$

is as well.

Remark 4.2. Let k be a perfect field in characteristic p. Define

$$K = \widehat{W(k)[1/p](\zeta_{p^n})}^{p\text{-adic}}$$

and

$$A = K^\circ\left\langle x_1^{1/p^\infty}, \ldots, x_n^{1/p^\infty}\right\rangle$$

Then the extension A_∞ is even faithfully flat over K°.

Proof. The idea is to use perfectoid geometry to almost calculate the rings B_ℓ. Then mod t, things simplify enormously.

Let $\ell \geq 1$. Apply the approximation lemma with $\epsilon = 1 - 1/p$ and $c = \ell$. Then there exists $f_\ell \in A\left\langle T^{1/p^\infty}\right\rangle^\flat$ with

$$|((T-g) - f_\ell^\sharp)(y)| \leq |t|^{1/p} \max(|(T-g)(y), |t|^\ell) \text{ for all } y \in Y.$$

Thus, we know that $f_\ell^\sharp \equiv T - g \mod t^{1/p}$, and that

$$Y\left(\frac{T-g}{t^\ell}\right) = Y\left(\frac{f_\ell^\sharp}{t^\ell}\right) \subseteq Y$$

We can almost calculate $B_\ell = \mathscr{O}_Y^+(Y(\frac{f_\ell^\sharp}{t^\ell}))$: by a lemma of Scholze,

$$\begin{aligned} B_\ell &\cong_a A\left\langle T^{1/p^\infty}\right\rangle \left\langle \left(\frac{f_\ell^\sharp}{t^\ell}\right)^{1/p^\infty} \right\rangle \\ &\cong_a \varinjlim_k \widehat{C_{\ell,k}}, \end{aligned}$$

where $C_{\ell,k} = A\left\langle T^{1/p^\infty}\right\rangle [u^{1/p^\infty}]/(ut^\ell)^{1/p^k} - (f_\ell^\sharp)^{1/p^k}$

It then suffices to show that $C_{\ell,k}/t$ is faithfully flat over A/t.

Lemma 4.3. *If R is a ring with $r \in R$ a nonzerodivisor, and M is an r-torsion-free R-module, then for all $n \geq 1$, we have that M/rM is faithfully flat over R/r if and only if M/r^nM is faithfully flat over R/r.*

If we apply the lemma to $R = A$, $M = C_{\ell,k}$, and $r = t^{1/p^{k+1}}$, we find that it's enough to show that $C_{\ell,k}/t^{1/p^{k+1}}$ is faithfully flat over $A/t^{1/p^{k+1}}$. But then $t^{1/p^k} \equiv 0$ in the quotient, so

$$C_{\ell,k}/t^{1/p^{k+1}} \cong A[T_1^{1/p^\infty}, u^{1/p^\infty}]/(t^{1/p^{k+1}}, (f_\ell^\sharp)^{1/p^k}).$$

But now we can form a diagram

$$\begin{array}{ccc} A/t^{1/p^{k+1}} & \xrightarrow{\quad\sim\quad} & A/t^{1/p} \\ \downarrow & & \downarrow \\ C_{\ell,k}/t^{1/p^{k+1}} & \xrightarrow{\sim} & A[T^{1/p^\infty}, u^{1/p^\infty}]/(t^{1/p}, T-g), \end{array}$$

where the horizontal arrows are Frob^k. But in fact the bottom right is free over the top right, so in particular, the map is faithfully flat.

□

This concludes the construction of A_∞.

References

[1] B. Bhatt, *On the direct summand conjecture and its derived variant*, preprint, 2017, arXiv: 1608.2v1 [math.AG].