THE RIEMANN EXTENSION THEOREM

KIRAN KEDLAYA

1. Disclaimer

This is the transcription of a lecture given by Kiran Kedlaya for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture describes André and Bhatt's perfectoid version of the Riemann Extension Theorem, used in the proof of the direct summand conjecture. We begin by giving Riemann's original formulation of the theorem, as well as a version for rigid analytic spaces, and then show how the results extend to the perfectoid case.

2. History

2.1. Riemann's Original Theorem.

Theorem 2.1.1 (Riemann)**.** *Let $U \subseteq \mathbf{C}$ be an open subset with $z_0 \in U$, and $f : U \setminus \{z_0\} \to \mathbf{C}$ a holomorphic function. If f is bounded on a punctured neighborhood of z_0 then f extends uniquely to a holomorphic function $\widetilde{f} : U \to \mathbf{C}$.*

Proof. Define $g : U \to \mathbf{C}$ by

$$g(z) = \begin{cases} f(z)(z - z_0^2) & z \neq z_0 \\ 0 & z = z_0 \end{cases}.$$

Then g is continuous and differentiable, and $g'(z_0) = 0$. By the Cauchy-Goursat theorem, g is holomorphic on U, so near z_0 we have

$$g(z) = \sum_{i=0}^{\infty} g_i (z - z_0)^i$$

with $g_0 = g(0) = 0$ and $g_1 = g'(0) = 0$. So set $\widetilde{f}$ to be the holomorphic function defined locally by

$$\widetilde{f} = \frac{g(z)}{(z - z_0)^2}.$$

□

Date: March 15, 2018.

[1]Typeset by Ashwin Iyengar

2.2. **Higher Dimensions.** The previous theorem was defined for open subsets of $\mathbf{C}$, but we can extend the result to any complex variety that is "nice enough": in particular, we need our variety to be normal, as we will show in a moment.

Theorem 2.2.1. *Let X be a normal complex variety, assume $g \in \mathscr{O}(X)$, and let g be a nonzerodivisor. Now let $Z := \{g = 0\}$, and say we have a holomorphic function $f : X \setminus Z \to \mathbf{C}$. If, for all compact subsets $K \subseteq Z$, f is bounded on some neighborhood of K in X, then there exists a unique holomorphic extension $\widetilde{f} : X \to \mathbf{C}$.*

The key here is really normality: for a counterexample, consider the vanishing locus $X = V(z_1 z_2)$ in the complex affine plane. Then take the function $z_1 + z_2$ on X, whose vanishing locus is just the origin $O = (0, 0)$, and suppose we have some holomorphic function $f : X \setminus O \to \mathbf{C}$, which is bounded on some disk around the origin intersected with X. Then the point is that we can't tell the difference between this space and its normalization, which is the disjoint union of two copies of $\mathbf{C}$, i.e. has coordinate ring $\mathbf{C}[x] \times \mathbf{C}[y]$. To see it explicitly, define the function

$$f(z_1, z_2) = \begin{cases} z_2 + 1 & z_1 = 0 \\ z_1 & z_2 = 0 \end{cases}$$

this is clearly holomorphic on $X \setminus O$ and bounded locally, but you clearly can't extend it to even a continuous function on X.

3. Non-Archimedean Case

This is due to Bartenwerfer.

Theorem 3.0.1. *Let K be a nonarchimedean field with $0 < |\varpi| < 1$, let A be a affinoid algebra over K, and suppose A is normal as a ring. Fix $g \in A$ a nonzerodivisor. Then*

$$A^\circ \xrightarrow{\sim} \varprojlim_n A \left\langle \frac{\varpi^n}{g} \right\rangle^\circ$$

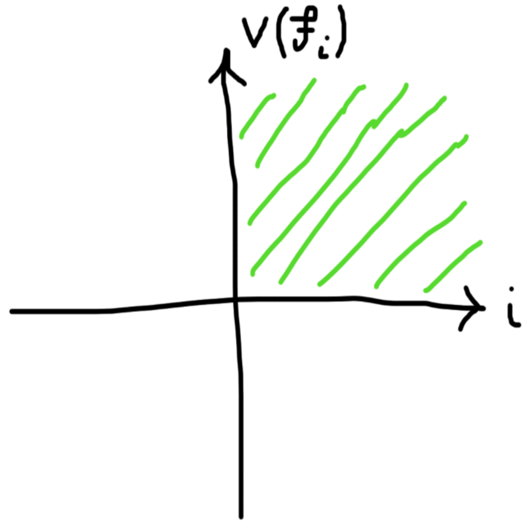
is an isomorphism.

We won't prove this in full generality, but we'll demonstrate why it should be true for the rigid analytic unit disc.

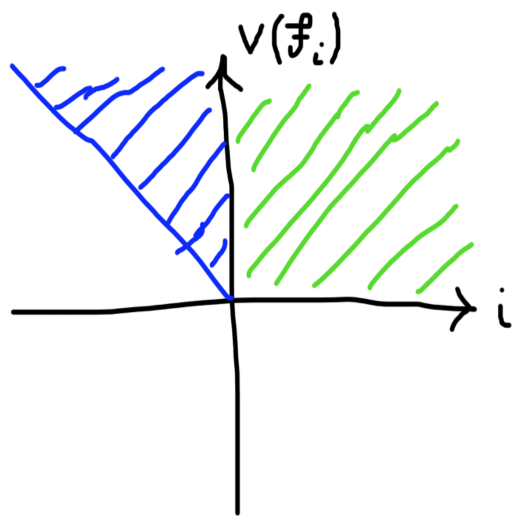
Example 3.0.2. Let $A = K\langle T \rangle$, and set $g = T$. Note the transition maps $A\langle \varpi^n/g \rangle^\circ \to A\langle \varpi^{n-1}/g \rangle^\circ$ are inclusions, and $A^\circ \subseteq A\langle \varpi^n/g \rangle^\circ$ and thus

$$\varprojlim_n A \left\langle \frac{\varpi^n}{g} \right\rangle^\circ = \bigcap_n \left(A \left\langle \frac{\varpi^n}{g} \right\rangle^\circ \right).$$

If we express the valuation additively (i.e. take a negative log) on K by v, then a given element in A° is a power series whose Newton polygon is contained in the green area in the following cartoon:



A given element in $A\langle \varpi^n/g\rangle^\circ$ can be expressed as a Laurent series in T, say $f = \sum_i f_i T^i$, where $f_i \in K$, with some size restriction on the f_i, which is pictorially expressed by saying that the Newton polygon of f is contained in the union of the green and blue areas in the following cartoon:



I n the above cartoon, the slope of the blue line is proportional to $-n$, so as n increases, the Newton polygons will increasingly be supported more towards the green quadrant. Taking the limit (i.e. intersection), one sees that anything that lives in the intersection can't be supported at any nonzero distance away from the y-axis in the blue area, hence the intersection must be exactly A°. This proves the isomorphism in this special case, but this is *not* a model for the general proof.

4. Almost commutative algebra in Pro-systems

Let R be a ring with $t \in R$ containing a compatible system of p^nth roots of t for all n. We want to work in t-almost mathematics, i.e. almost mathematics with respect to the ideal (t^{1/p^∞}). Define proMod_R to be the category of pro-systems of R-modules, indexed by $\mathbf{N}$.

Definition 4.0.1. A system $\{M_n\}_{n>0} \in \mathsf{proMod}_R$ is almost-pro-zero if

$$\forall k, \forall n > 0, \exists n' \geq n \text{ such that: } t^{1/p^k} \operatorname{im}(M_{n'} \to M_n) = 0.$$

Remark 4.0.2 (Warning)**.** Saying $\{M_n\}_{n>0}$ is zero in proMod_R^a is the same as saying

$$\forall n, \exists n' \geq n \text{ such that: } \forall k, t^{1/p^k} \operatorname{im}(M_{n'} \to M_n) = 0.$$

which is different. In particular zero in proMod_R^a implies almost-pro-zero, but not the converse.

Some facts:

(1) If $\{M_n\}_{n>0}$ is almost-pro-zero, then

$$(\mathbf{R}\varprojlim_n) M_n = 0 \text{ in } \mathsf{DMod}_R^a,$$

the derived category of Mod_R^a. This is because for all k, the map

$$\{M_n[t^{1/p^k}]\}_{n>0} \to \{M_n\}_{n>0}$$

gives a pro-isomorphism.

(2) Almost-pro-zero systems remain almost-pro-zero under any R-linear functor.

5. The Perfectoid Riemann Extension Theorem

Scholze first proved a version of this in [2] and Bhatt in [1].

Theorem 5.0.1. *If K is a perfectoid field, choose a pseudo uniformizer $0 < |\varpi| < 1$ of the form $\varpi = \varpi_0^\sharp$ for $\varpi_0 \in K^\flat$. Let A be a perfectoid K-algebra, and fix $g \in A^\circ$ of the form $g_0^\sharp$ for $g_0 \in A^{\flat\circ}$. Choose $m \geq 0$. Then the pro-morphism $f : \{A^\circ/\varpi^m\}_{n>0} \to \{A \langle \varpi^n/g \rangle^\circ /\varpi^m\}_{n>0}$ given by*

$$\{f_n : A^\circ/\varpi^m \to A \left\langle \frac{\varpi^n}{g} \right\rangle^\circ /\varpi^m\}_{n>0}$$

is a (ϖg)-almost-pro-isomorphism, i.e. the kernel and cokernel of this map is (ϖg)-almost-pro-zero in $\mathsf{proMod}_{A^\circ}$.

Remark 5.0.2. (1) We can work in ϖg-almost mathematics because g is assumed to be an untilt, and thus has a compatible system of p-power roots.

(2) Note in particular that we don't require any normality condition, which is surprising, and says that something is really different in this perfectoid situation. We will remark further on this in the next example. We should mention that A perfectoid implies A seminormal.

(3) Futhermore we don't require that g is a nonzerodivisor. In fact, one can show that
$$\ker\left(A^\circ \to \varprojlim_n A\left\langle \frac{\varpi^n}{g}\right\rangle^\circ\right)$$
is almost zero.

(4) This should imply the previous theorem of Bartenwerfer.

Example 5.0.3. Let's consider the perfectoid version of our non-normal counterexample to the Riemann extension theorem. Fix your favorite perfectoid field K and take $A = K\left\langle T_1^{1/p^\infty}, T_2^{1/p^\infty}\right\rangle/\overline{(T_1T_2)^{1/p^\infty}}$, where the bar denotes topological closure. As before, let $g = T_1 + T_2$ and consider the map
$$A^\circ \to \varprojlim_n A\left\langle \frac{\varpi^n}{g}\right\rangle^\circ.$$
This map is clearly injective, so it suffices to show that the cokernel is killed by the ideal $((\varpi g)^{1/p^\infty})$. But
$$\varprojlim_n A\left\langle \frac{\varpi^n}{g}\right\rangle^\circ \cong K^\circ\left\langle T_1^{1/p^\infty}\right\rangle \times K^\circ\left\langle T_2^{1/p^\infty}\right\rangle,$$
and the cokernel of the above map is the difference of the constant terms of the two Tate algebras, which lands in K°. But given a pair $(f, g) \in K^\circ\left\langle T_1^{1/p^\infty}\right\rangle \times K^\circ\left\langle T_2^{1/p^\infty}\right\rangle$, one can check that multiplication by g^{1/p^k} kills the constant term.

Example 5.0.4. As a second example, let K be as before, and then let $A = K\left\langle T^{1/p^\infty}\right\rangle$ be the perfectoid affine line. We take $g = T$. Then we can show, via a similar picture to the first picture, that
$$K^\circ\left\langle T^{1/p^\infty}\right\rangle/\varpi^m \xrightarrow{\sim} \varprojlim_n K\left\langle T^{1/p^\infty}\right\rangle\left\langle \frac{\varpi^n}{T}\right\rangle^\circ/\varpi^m.$$
But the condition of the pro-morphism f being an almost-pro-isomorphism is actually stronger! We'll draw another picture in a moment to illustrate how to prove the theorem in this case.

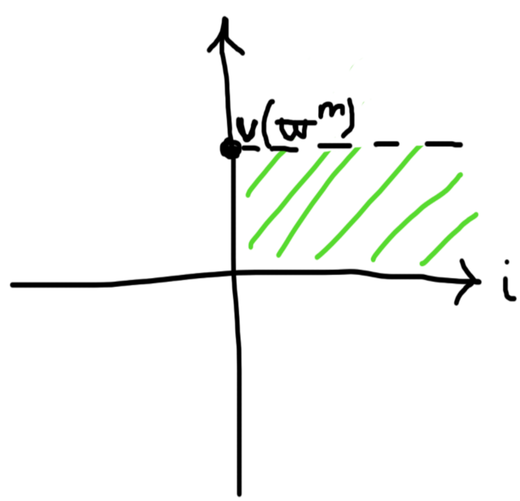
In fact, Example 5.0.4 is all we need to deduce the theorem in full generality: to see this, note that we can tensor the maps
$$\{f_m : K^\circ\left\langle T^{1/p^\infty}\right\rangle/\varpi^m \to K^\circ\left\langle T^{1/p^\infty}\right\rangle\left\langle \frac{\varpi^n}{T}\right\rangle^\circ/\varpi^m\}_{n>0}.$$
However, given a perfectoid algebra A over K and a g as in the Theorem, we can define a map $K^\circ\left\langle T^{1/p^\infty}\right\rangle \to A^\circ$ which sends $T^{1/p^\infty} \mapsto g^{1/p^\infty}$. If we tensor this map with the above pro-isomorphism, one can show that in fact the targets of the f_m become ϖ-almost the pro-morphism you wanted to show in the general case. It's remarkable that the almost mathematics is able to reduce the entire theorem to what is essentially the simplest example.

Finally, we will prove the case of the simplest example, which by the above discussion, proves the theorem.

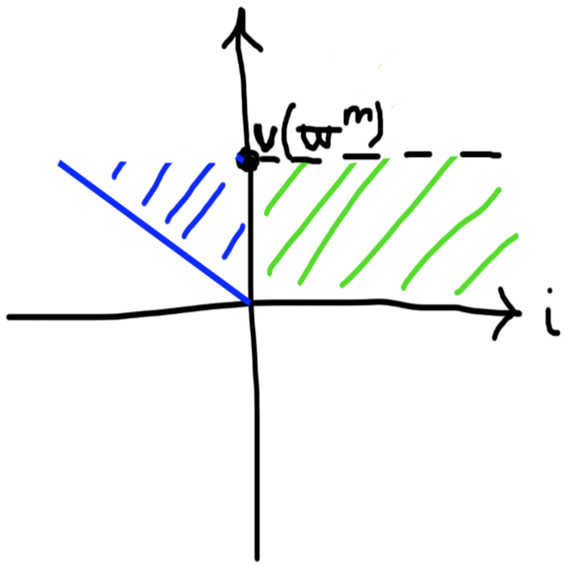
Proof of Example 5.0.4. For $A = K\left\langle T^{1/p^\infty}\right\rangle$ and $g = T$, we want to check that the pro-morphism

$$\{f_m : K^\circ \left\langle T^{1/p^\infty}\right\rangle/\varpi^m \to K^\circ\left\langle T^{1/p^\infty}\right\rangle \left\langle \frac{\varpi^n}{T}\right\rangle^\circ/\varpi^m\}$$

is an almost-pro-isomorphism. It suffices to show that the cokernel is almost-pro-zero, so we need to show that for each k and n, we can find n' such that $\operatorname{coker}(f_{n'}) \to \operatorname{coker}(f_n)$ is killed by $(\varpi T)^{1/p^k}$. In fact we'll only use T^{1/p^k}. Now, given $f \in K^\circ\left\langle T^{1/p^\infty}\right\rangle/\varpi^m$, we can pick a minimal representative so that the support diagram lies in the green region in the following cartoon:



In other words, it lies below the (additive) valuation of ϖ^m for each monomial T^a, where $a \in \mathbf{Z}_{(p)}$. Now an element in $K^\circ\left\langle T^{1/p^\infty}\right\rangle\left\langle\frac{\varpi^n}{T}\right\rangle^\circ/\varpi_m$ can be represented by something that lies in the blue or green area in the following cartoon:



∎

Multiplying by some T^a for $a \in \mathbf{Z}_{(p)}$ moves the blue area into the green area, but if we fix a small k and large n, it may not shift all the blue into all the green, which would kill the cokernel as desired. This is why we need the image of a higher n: choosing a higher n would make the slope of the blue line much steeper, which then controls the total horizontal width of the blue area. If we choose n large enough, then the horizontal width of the blue area will eventually be less than $p^{1/k}$, which is what we want. This concludes the proof.

References

[1] B.Bhatt, *On the direct summand conjecture and its derived variant*, preprint, 2017, arXiv: 1608.2v1 [math.AG].

[2] P. Scholze, *On torsion in the cohomology of locally symmetric spaces*, Ann. of Math. (2) 182 (2015), no. 3, 945-1066.