THE DIRECT SUMMAND CONJECTURE AND ITS DERIVED VARIANT

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1. Disclaimer

This is the transcription of a lecture given by Bhargav Bhatt for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture describes Bhatt's proof of the direct summand conjecture.

2. Overview

The goal of the talk is to explain the following theorem:

Theorem 2.1 (Andre)**.** *Let R be a regular ring, let $R \to S$ be a finite ring extension. Then $R \to S$ splits in Mod_R.*

The rough strategy of the proof is as follows:

(1) Construct a huge ring extension $R \to A$ which is almost faithfully flat with A perfectoid.

(2) Now try to show that $A \to S \otimes_R A$ "almost" splits. This will use the Riemann Extension Theorem, and thus the notion of almost is different than in the first step.

(3) Finally, descend the splitting.

Using essentially the same strategy, one can prove the following derived version:

Theorem 2.2. *Let R be a regular ring, $f : X \to \operatorname{Spec} R$ proper surjective. Then $R \to \mathbf{R}\Gamma(X, \mathscr{O}_X)$ splits in $D\mathsf{Mod}_R$.*

The strategy is roughly the same, but to get the splitting, you also need the Tate acyclicity theorem.

Exercise 2.3. *Take R regular, and $X \to \operatorname{Spec} R$ is proper and birational. Prove Theorem 2.0.2 in this case.*

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[1]Typeset by Ashwin Iyengar

3. Finding Perfectoid Covers

Fix a prime p, and a perfectoid field $K = \widehat{\mathbf{Q}_p(p^{1/p^\infty})}$.

Proposition 3.1 (R)**.** *Say R is a regular ring, $0 \neq p$, and p belongs to the Jacobson radical of R. Then there exists a perfectoid affinoid K-algebra $(A[1/p], A)$ along with a map $R \to A$ which is almost faithfully flat (see below).*

Remark 3.2. (1) There is a subtlety here: we only defined almost faithfully flat over something where we can actually talk about almost mathematics. In particular, there is no notion of this for R a priori, but we can talk about almost faithful flatness for $R \to A$ by noticing that if M is an R-module, then $\mathrm{Tor}_i^R(A, M)$ is an A-module, so almost flatness can be described by saying that $\mathrm{Tor}_i^R(A, -)$ is almost zero for all $i > 0$.

(2) We can drop "almost" with a more liberal notion of "perfectoid".

(3) We can also assume that any $g \in R$ admits a compatible sysmte of p-power roots in A.

Now we state some example cases:

Example 3.3. (1) Take $R = \widehat{\mathbf{Z}_p[x_2, \ldots, x_d]}$. Then we can take

$$A = \widehat{R[p^{1/p^\infty}, x_i^{1/p^\infty}]}$$

(2) If $R = \mathbf{Z}_p[[x_1, \ldots, x_d]]/(p - f(x_i))$, where $f \in (x_i)^2$ and p doesn't divide x, then this is still a regular local ring, but now we can take

$$B = \widehat{R[x_i^{1/p^\infty}]}$$

and then set A to be the p-adic completion of the integral closure of $B[p^{1/p^\infty}]$ in $B[p^{1/p^\infty}, 1/p]$.

4. DSC (unramified in characteristic 0)

Again, let $f : R \hookrightarrow S$ be a finite extension with R regular and p-complete. Then I want to assume that $f[1/p]$ is étale (for example, you could take $S = R[u^{1/p^\infty}]$ for $u \in R^\times$).

Then the goal is to show:

Theorem 4.1 (U)**.** *$R \to S$ splits.*

Proof. If you want to prove that something splits, you should consider the canonical obstruction class to this splitting. In particular, there is a short exact sequence

$$0 \to R \to S \to S/R \to 0,$$

in Mod_R, which defines an obstruction class $\mathrm{ob} \in \mathrm{Ext}^1_R(S/R, R)$. Now choose $R \to A$ as in Proposition R. Now set B to be the integral closure of $S \otimes_R A$ inside $S \otimes_R A[1/p]$. This gives a diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

We want to split $R \to S$, but we know that $R \to A$ is (almost) faithfully flat, so it's "almost" enough to split $A \to B$, in some precise sense (note the integral closure here).

Note we had a crucial assumption that $f[1/p]$ was étale, so $R \to S$ is étale on the generic fibers, and thus $A \to B$ is as well. So by the almost purity theorem, then B is almost finite étale over A. This implies that $A \to B$ is almost split, in the sense that the obstruction is almost 0.

Thus $A \to S \otimes_R A$ is also almost split, so $\mathrm{ob} \otimes 1 \in \mathrm{Ext}^1_A(S/R \otimes A, A)$ is almost zero. Really there's some subtlety here because of "almost faithful flatness", but we'll ignore it.

Now, as $R \to A$ is almost faithfully flat, if we take

$$\mathrm{Ext}^1_R(S/R, R) \otimes_R A \cong_a \mathrm{Ext}^1_A(S/R \otimes A, A),$$

sending $\mathrm{ob} \otimes 1 \mapsto \mathrm{ob}$, which is almost zero. But now this shows us that $\mathrm{ob} \otimes 1$ is almost zero. So now we want to go back down.

We also have $\mathrm{Ann}_R(\mathrm{ob})A =_a \mathrm{Ann}_A(\mathrm{ob})$, and thus we want to show that $\mathrm{Ann}_R(\mathrm{ob})A = A$. But we know that $\mathrm{Ann}_A(\mathrm{ob})$ is very large, since ob (for $A \to B$) is almost zero. On the other hand, Ann_R is an ideal in a Noetherian ring, so it's finitely generated, but after pushing it forward to A you get something extremely huge, which forces $\mathrm{Ann}_R(\mathrm{ob})$ to be very large.

One way to flesh this out is the following. Because $\mathrm{Ann}_A(\mathrm{ob})$ is almost zero, we have $p^{1/p^n} \in \mathrm{Ann}_A(\mathrm{ob})$ for all n. From this, you can show that $p^2 \in \mathrm{Ann}_R(\mathrm{ob})^{p^n}$ for all n. But now Krull's theorem tells you that this can't happen unless $\mathrm{Ann}_R(\mathrm{ob}) = (1)$. So $\mathrm{ob} = 0$, and we're done. □

Remark 4.2. So far this uses technology we had before 2012. For the ramified case, we need more modern stuff.

5. DSC (in general)

Again, $f : R \hookrightarrow S$ is a finite extension, R is regular and p-complete, and $p \neq 0$.

Theorem 5.1 (A)**.** *$f : R \to S$ splits.*

Proof. Choose $g \in R$ such that $f[1/pg]$ is étale. Now choose $R \to A$ as in Proposition R such that p, g admit a compatible system of p-power roots in A.

Note p doesn't play a special role in Theorem U, so we can just repeat the argument with pg. Thus it suffices to show that $A \to S \otimes_R A$ is pg-almost split, and plug it in to the other argument.

Now consider $A\langle p^n/g\rangle \xrightarrow{f_n} S \otimes_R (A\langle p^n/g\rangle)$ for all n. The key observation is that $f_n[1/p]$ is étale, since g divides p^n in $A\langle p^n/g\rangle$ by construction. Again, the ring $A\langle p^n/g\rangle$ is a perfectoid algebra, $S \otimes_R A\langle p^n/g\rangle$ is étale after inverting p, so almost purity tells you that f_n is p-almost split.

Also, we have

$$\mathrm{Ext}^1_A(S/R \otimes_R A, A) \to \varprojlim_n \mathrm{Ext}^1_{A\langle p^n/g\rangle}(S/R \otimes A\langle p^n/g\rangle, A\langle p^n/g\rangle).$$

But in each of the Ext's in the limit, the obstruction is p-almost zero. We know that the above map is a (pg)-almost isomorphism (because of the Riemann Extension Theorem and the fact that Ext is R-linear), so combining these, we get that the original obstruction is pg-almost zero.

Note, to do this properly you really need to do with mod powers of p, as in Kedlaya's lecture on the Riemann Extension Theorem, but we've shoved that under the rug.

Also you need to work with the derived tensor product in order to really use that Ext respects base change nicely. $\square$

6. Derived DSC

Let R be a regular ring with $0 \neq p$, and p in the Jacobson radical of R. Fix $f : X \to \operatorname{Spec} R$ proper surjective. Then

Theorem 6.1. *$R \to R\Gamma(X, \mathscr{O}_X)$ splits in $D\mathsf{Mod}_R$.*

Proof. We treat the case f is birational (and then need to do the case of f finite somewhere else). As f is birational, choose $g \in R$ such that $f[1/pg]$ is an isomorphism. Now choose $R \to A$ as before. We have $X \xrightarrow{f} \operatorname{Spec} R$. Now take

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"] & X_A \arrow[r] \arrow[d, "f_A"] & X \arrow[d, "f"] \\ \operatorname{Spec} A\langle p^n/g\rangle \arrow[r] & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

By almost faithful flatness, we can show f_A satisfies the theorem. By tracing through the argument some more, it suffices to show that

$$A\langle p^n/g\rangle \to \mathbf{R}\Gamma(X_n, \mathscr{O}_{X_n})$$

is almost split.

Now as $f_n[1/p]$ is an isomorphism, we get a lift

$$\begin{tikzcd}
S_\eta^+ := (\mathrm{Spa}(A\left\langle p^n/g\right\rangle[1/p], A\left\langle p^n/g\right\rangle), \mathscr{O}^+) \arrow[r, dashed, "!"] \arrow[dr] & X_n \arrow[d, "f_n"] \\
& \mathrm{Spec}\, A\left\langle p^n/g\right\rangle =: S
\end{tikzcd}$$

which comes from the valuative criterion of properness.

Now apply $\mathbf{R}\Gamma(-, \mathscr{O})$ to this diagram, and one gets

$$\begin{tikzcd}
A\left\langle p^n/g\right\rangle \arrow[r] \arrow[dr] & \mathbf{R}\Gamma(X_n, \mathscr{O}_{X_n}) \arrow[d] \\
& \mathbf{R}\Gamma(S_\eta^+, \mathscr{O}^+)
\end{tikzcd}$$

But the diagonal arrow is an almost isomorphism. and thus we get the almost splitting this way. □

References

[1] B. Bhatt, *On the direct summand conjecture and its derived variant*, preprint, 2017, arXiv: 1608.2v1 [math.AG].