ALMOST PURITY

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1. Disclaimer

This is the transcription of a lecture given by Yves André for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture states and gives a very rough sketch of the proof of almost purity theorem in general characteristic.

2. Statement

Let K be a perfectoid field, $\mathfrak{m} = K^{\circ\circ} \subseteq K^{\circ} \subseteq K$ be the maximal ideal in the ring of power bounded elements in the field, and $\varpi \in \mathfrak{m}$ be a pseudo-uniformizer satisfying $|p| < |\varpi| < 1$.

2.1. **Theorem.** *Let $\mathcal{A}$ be a perfectoid K-algebra, and let $\mathcal{B}$ be a finite étale $\mathcal{A}$-algebra. Then*

(1) $\mathcal{B}$ is perfectoid.

(2) (almost purity) $\mathcal{B}^{\circ}$ is an almost finite étale $\mathcal{A}^{\circ}$-algebra.

Furthermore, if $\mathcal{A}_{\text{fét}}$ is the category of finite étale $\mathcal{A}$-algebras, then tilting gives an equivalence of categories bewteen

$$A_{\text{fét}} \cong A^{\flat}_{\text{fét}}.$$

Tate proved this for some perfectoid fields. Faltings proved this for some (many) perfectoid algebras, and it was proved in full generality by Scholze and Kedlaya-Liu independently.

3. Reduction to the Galois case

We will show $(i) \implies (ii)$. The idea is reduce to the Galois case, i.e. the case where $\mathcal{B}^G = \mathcal{A}$ for some finite group G acting on $\mathcal{B}$, and

$$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \prod_G \mathcal{B}$$

sending $b \otimes b' \mapsto (g(b)b')$ is an isomorphism. It's well known that if $\mathcal{B}$ is G-Galois over $\mathcal{A}$, then $\mathcal{B}$ is finite étale over $\mathcal{A}$. The idea is that the isomorphism gives you some idempotent

Date: March 12, 2018.

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element in $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$, which kills the kernel of multiplication $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$, which you can then use to show that $\mathcal{B}$ is finite projective over $\mathcal{A}$, from which we can reduce the result. An analogous result holds in the almost setting, as noted in [1].

Now we reduce the Galois situation. After decomposing A, one can assume that the rank $[B : A] = n$ is constant. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and $Z = Y \times_X \cdots \times_X Y \setminus D$ where D is the partial diagonals, then in fact $Z = \operatorname{Spec} C$ for some C, which is S_n-Galois over X and S_{n-1}-Galois over Y, both via the first projection.

Then an application of faithfully flat descent reduces us to the Galois case fully.

Now assume (i). By assumption we have $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \cong \prod_G \mathcal{B}$, and $\mathcal{B}$ is perfectoid, so $\mathcal{B}\otimes_{\mathcal{B}}$ is perfectoid as well. Additionally, the map

$$\mathcal{B}^\circ\widehat{\otimes}_{\mathcal{A}^\circ}\mathcal{B}^\circ \xrightarrow{\sim} (\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B})^\circ$$

is an almost isomorphism. But by the decomposition we get

$$\mathcal{B}^\circ\widehat{\otimes}_{\mathcal{A}^\circ}\mathcal{B}^\circ \xrightarrow{\sim} (\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B})^\circ = \prod_G \mathcal{B}^\circ,$$

So we almost have what we want, except for the completion in the tensor product. But we can get around this by taking a quotient by ϖ, and one can deduce that B°/ϖ is almost Galois over A°/ϖ, hence almost finite étale. Finally, one can deduce that $\mathcal{B}^\circ$ is finite étale over A°.

4. Proof of (i)

The proof is done in seven steps. We will sketch, very roughly, each one.

(1) First, we look at perfectoid fields. But this is already done: if $\mathcal{A}$ is a field, then $G_{\mathcal{A}} \cong G_{\mathcal{A}^\flat}$, so finite extensions of $\mathcal{A}$ correspond to finite extensions of $\mathcal{A}^\flat$, and one can show that since extensions of $\mathcal{A}^\flat$ are perfectoid, extensions of $G_{\mathcal{A}}$ are as well.

(2) For perfectoid algebras in characteristic $p > 0$, proving perfectoid-ness is equivalent to proving that $\mathcal{B}$ is perfect and that $\mathcal{B}^\circ$ is bounded. Since $\mathcal{A}$ is reduced, $\mathcal{B}$ is also reduced. Thus $\Omega_{\mathcal{B}/\mathcal{A}} = 0$, so $\Omega_{\mathcal{B}/\mathcal{A}[\mathcal{B}^p]} = 0$. But any element c in the kernel of the map $\mathcal{B} \otimes_{\mathcal{A}[\mathcal{B}^p]} \mathcal{B} \xrightarrow{\mu} \mathcal{B}$ has the property that $c^p = 0$. But $\Omega_{\mathcal{B}/\mathcal{A}[\mathcal{B}^p]}$ is this kernel mod itself squared, and one can deduce that

$$\mathcal{B} \otimes_{\mathcal{A}[\mathcal{B}^p]} \mathcal{B} \xrightarrow{\sim} \mathcal{B}$$

is an isomorphism. By a lemma in EGA, one can deduce that $\mathcal{A}[\mathcal{B}^p] \to \mathcal{B}$ is surjective. This shows that the Frobenius is surjective, so $\mathcal{B}$ is perfect.

Now we need to show that $\mathcal{B}^\circ$ is bounded. We define $\mathcal{B}_0$, a finite sub-$\mathcal{A}^\circ$-algebra of $\mathcal{B}$ such that $\mathcal{B} = \mathcal{B}_0[1/\varpi]$. Then $\mathcal{B}_0$ is contained in the integral closure of $\mathcal{A}^\circ$ in $\mathcal{B}$, which is in turn contained in

$$\mathcal{B}_0' = \{b \in \mathcal{B} : \operatorname{tr}_{\mathcal{B}/\mathcal{A}}(b\mathcal{B}_0) \subseteq \mathcal{A}^\circ\}.$$

One has

$$\mathrm{Ann}(\mathcal{B}_0'/\mathcal{B}_0)[1/\varpi] = \mathcal{B},$$

and there is some power n such that $\varpi^n \mathcal{B}_0' \subseteq \mathcal{B}_0$.

(3) Now look at characteristic 0. Given $\mathcal{B}^\flat \in \mathcal{A}^\flat_{\text{fét}}$, we have $\mathcal{B} \in \mathcal{A}_{\text{fét}}$. To see this, look at

$$\mathcal{B}^{\flat\circ}/\varpi^\flat \cong \mathcal{B}^\circ/\varpi.$$

But to check finite étale-ness, it's enough to check it mod ϖ, and then we can just use almost purity in characteristic p.

(4) Now we introduce perfectoid spaces. Take $X = \mathrm{Spa}(\mathcal{A}, \mathcal{A}^\circ)$, $Y = \mathrm{Spa}(\mathcal{B}, \mathcal{B}^\circ)$ (so we get a finite étale map $Y \to X$), and the tilt $X^\flat$, defined as usual. Now we want to show that Y is perfectoid.

Pick $x \in X$ with corresponding point $x^\flat \in X^\flat$. We have $\mathscr{O}_{X,x}$ containing $\mathscr{O}^+_{X,x}$, which are both Henselian, and the completed residue field $\kappa(x)$ containing the completed residue field $\kappa(x)^+$. Note $\kappa(x)$ is perfectoid, and its tilt is $\kappa(x^\flat)$. Furthermore,

$$\ker(\mathscr{O}^+_{X,x} \to \kappa(x)^+)$$

is ϖ-divisible. Then

$$\widehat{\mathscr{O}^+_{X,x}}[1/\varpi] = \kappa(x).$$

(5) This is the Henselian approximation step.

4.1. **Proposition.** *If R is a flat K°-algebra, Henselian along ϖ, then*

$$R[\varpi^{-1}]_{\textit{fét}} \xrightarrow{\sim} (\widehat{R}[\varpi^{-1}])_{\textit{fét}}$$

is an equivalence of categories.

4.2. **Remark.** Consider a smooth R-algebra S, and let $Z = \operatorname{Spec} S$. By the infinitesimal lifting property of a smooth map, that

$$Z(R) \to Z(R/\varpi)$$

is surjective. This is also true in the Henselian case: embed $Z \subseteq \mathbf{A}^n$ and construct the normal bundle to this embedding. Then a neighborhood of the zero section will be étale over its image in $\mathbf{A}^n$.

For example, there is a bijection on isomorphism classes $\mathsf{fProj}_R \to \mathsf{fProj}_{R/\varpi}$. One can deduce from this that there is a bijection on isomorphism classes of $R_{\text{fét}} \to (R/\varpi)_{\text{fét}}$.

But now only assume that $S[\varpi^{-1}]$ is smooth over $R[\varpi^{-1}]$. Then you can a approximate formal solution by real solutions: this is due to Elkik, which uses Newton's lemma.

Then you get a uniform Mittag-Leffler condition for approximation mod powers of ϖ.

In any case, apply the proposition to $R = \mathscr{O}^+_{X,x}$, to obtain

$$(\mathscr{O}^+_{X,x}[\varpi^{-1}])_{\text{fét}} \xrightarrow{\sim} \kappa(x)_{\text{fét}}.$$

But the left hand side is a 2-colimit (over rational subsets U containing x) of $\mathscr{O}_X(U)_{\text{fét}}$. Additionally, we have the same equivalence in the tilted case, between $\kappa(x^\flat)$ and the 2-colimit of the $(\mathscr{O}_{X^\flat}(U^\flat))_{\text{fét}}$ for $U^\flat$ containing $x^\flat$.

(6) Now we untilt, by taking the 2-colimit over the $U^\flat$ containing $x^\flat$ of $\mathscr{O}_{X^\flat}(U^\flat)_{\text{fét}}$, then untilting each object in the result colimit category, so we get objects which are perfectoid in characteristic 0. Now by step 3, we have

$$\mathscr{O}_{X^\flat}(U^\flat)^\sharp_{\text{fét}} \subseteq (\mathscr{O}_X(U))_{\text{fét}}.$$

Now fix $\mathcal{B} \in \mathcal{A}_{\text{fét}} = \mathscr{O}_X(X)_{\text{fét}}$, and define $\mathcal{B}|_U = \mathcal{B} \otimes_{\mathcal{A}} \mathscr{O}_X(U) \in \mathscr{O}_X(U)_{\text{fét}}$. Then by cofinality, there exists $U^\flat$ such that $\mathcal{B}|_U \cong (\mathscr{O}_{X^\flat}(V^\flat))^\sharp$ for some $V^\flat$ finite étale over $U^\flat$.

by using quasi compactness of X, we can find a finite cover $X = \bigcup U_i$ such that

$$\mathcal{B} \otimes_{\mathcal{A}} \mathscr{O}_X(U_i) = \mathscr{O}_X(V_i^\flat)^\sharp$$

for some $V_i^\flat \in (U_i^\flat)_{\text{fét}}$.

(7) The last step is gluing. We can glue the $V_i^\flat$ to get an affinoid perfectoid $V^\flat$ over $X^\flat$. For this, we need Noetherian approximation.

Now we untilt to get $Y \supseteq X \subseteq V$. Both Y and V are finite étale, and V is perfectoid, so we need to show that $Y = V$ to show that Y is perfectoid. For this, we appeal to Tate acyclicity for X and V.

For X, we get

$$\mathcal{A} \to \bigoplus_i \mathscr{O}_X(U_i) \to \bigoplus_{i<i'} \mathscr{O}_X(U_i \cap U_{i'}) \to \cdots$$

Tensoring with $\mathcal{B}$ over $\mathcal{A}$ and using Tate acyclicity for V, we get

$$\begin{array}{ccccccc}
\mathcal{B} & \longrightarrow & \bigoplus_i \mathcal{B} \otimes_{\mathcal{A}} \mathscr{O}_X(U_i) & \longrightarrow & \bigoplus_{i<i'} \mathcal{B} \otimes_{\mathcal{A}} \mathscr{O}_X(U_i \cap U_{i'}) & \longrightarrow & \cdots \\
\| & & \| & & \| & & \\
\mathscr{O}(V) & \longrightarrow & \bigoplus_i \mathscr{O}_V(V_i) & \longrightarrow & \bigoplus_{i<i'} \mathcal{B} \otimes_{\mathcal{A}} \mathscr{O}_V(V_i \cap V_j) & \longrightarrow & \cdots
\end{array}$$

where the second two columns are isomorphic, so we get $B = \mathscr{O}_V$, and thus $\mathcal{B}$ is perfectoid.

This concludes the proof. To finish, we give a non-example, for $p = 2$.

Let $K = \mathbf{Q}_2(\mu_{2^\infty})$. Let

$$\mathcal{A}^\circ = \widehat{\bigcup_k} \mathbf{Z}_2[\mu_2^k][[x^{2^{-k}}]]$$

and $\mathcal{A} = \mathcal{A}^\circ[1/2]$, which is perfectoid over K. Then let

$$\mathcal{B} = \mathcal{A}[\sqrt{x^2+4},$$

which is finite and ramified over $\mathcal{A}$. The claim is that $\mathcal{B}$ is not perfectoid. For instance,

$$\frac{i+1}{2}(\sqrt{x^2+4}-\sqrt{x}) \in B^\circ,$$

but has no square root mod 2.

References

[1] Y. André, *La lemme d'Abhyankar perfectoide*, preprint, 55 pp. arXiv:1609.00320 [math.AG] August 31, 2016.