PERFECTOID MULTIPLIER/TEST IDEALS AND SYMBOLIC POWERS

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1. Disclaimer

This is the transcription of a lecture given by Linquan Ma for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture first describes the construction and use of multiplier ideals and test ideal in pure characteristic 0 or p, then is followed by a discussion of a new definition of a multiplier/test ideal in mixed characteristic $(0, p)$. The new definition involves perfectoid techniques. This is joint work with Karl Schwede.

2. Motivation

We start with the following motivating theorems.

Theorem 2.1 (Ein-Lazarsfeld-Smith [3], Hochster-Huneke [5])**.** *If R is a Noetherian regular ring containing a field and $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{q} = h$, then*

$$\mathfrak{q}^{(hn)} \subseteq \mathfrak{q}^n,$$

for all n, where $\mathfrak{q}^{(nh)}$ is the nh symbolic power of $\mathfrak{q}$. In particular, if $\dim R = d \leq \infty$, then

$$\mathfrak{q}^{(dn)} \subseteq \mathfrak{q}^n$$

for all $\mathfrak{q}$ and all n.

Theorem 2.2 (Swanson [8])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I \subseteq R$ be an ideal in R such that the I-adic topology is equivalent to the topology defined by the ideals $\{I^n : \mathfrak{m}^\infty\}_n$. Then there exists an integer ℓ such that for all $n \geq 1$,*

$$I^{\ell n} : \mathfrak{m}^\infty \subseteq I^n.$$

Theorem 2.1 was surprising, because d is independent of $\mathfrak{q}$. But we can ask:

(1) Does Theorem 2.1 hold for general regular rings?

(2) If $(R, \mathfrak{m})$ is a complete local domain, then does there exist ℓ such that $I^{ln} : \mathfrak{m}^\infty \subseteq I^n$ for *every* $I \subseteq R$ and every $n \geq 1$?

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[1]Typeset by Ashwin Iyengar

Partial answers to (2) are known, but very little is known in general. For example, (2) is known when R is an isolated singularity in equal characteristic as in [6], but ℓ is not explicit in this result.

The goal of this talk, however, is to focus on (1). The main result is

Theorem 2.3 (Ma-Schwede [7])**.** *(1) is true when R is excellent.*

The strategy of the proof in [3] is to use multiplier ideals and vanishing theorems, in characteristic 0. The proof in [5] uses the tight closure in characteristic $p > 0$, followed by a reduction mod p technique along with Artin approximation. There's also a proof given by Hara in [4], where the strategy is to replace the multiplier ideal by something called a "test ideal" in characteristic $p > 0$, which will be revealed soon.

3. Multiplier/Test Ideals

Definition 3.1. Let $(R, \mathfrak{m})$ be a regular local ring, essentially of finite type of k of characteristic 0. Take an ideal $\mathfrak{a} \subseteq R$ and $t \in \mathbf{R}_{\geq 0}$. Then the multiplier ideal associated to $\mathfrak{a}$ and t is defined as

$$J(R, \mathfrak{a}^t) = \pi_* \mathscr{O}_Y(K_{Y/X} - \lfloor t\mathcal{G} \rfloor),$$

where $Y \to X = \operatorname{Spec} R$ is a log resolution with respect to $\mathfrak{a}$, where $\mathfrak{a}\mathscr{O}_Y = \mathscr{O}_Y(-\mathcal{G})$.

Remark 3.2. It turns out that this is independent of the log resolution $Y \to \operatorname{Spec} R$.

Example 3.3. If $R = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, and $\mathfrak{a} = (x_1^{a_1} \ldots x_d^{a_d})$, then

$$J(R, \mathfrak{a}^t) = (x_1^{\lfloor a_1 t \rfloor}, \ldots, x_d^{\lfloor a_d t \rfloor}).$$

Definition 3.4. Now in positive characterstic, suppose $(R, \mathfrak{m})$ is a regular local ring over k which has characteristic p, and suppose R is F-finite (i.e. Frobenius finite). Again let $\mathfrak{a} \subseteq R$ be an ideal and $t \in \mathbf{R}_{\geq 0}$. Then the test ideal

$$\tau(R, \mathfrak{a}^t) = \sum_{e >> 0} \operatorname{Tr}^e(c^{1/p^e} (\mathfrak{a}^{\lceil tp^e \rceil})^{1/p^e})$$

where $\operatorname{Tr}^e : R^{1/p^e} \to R$ is the trace map. The c here is the "test element", which, for R regular, can be any nonzero element. Again, one can show that $\tau(R, \mathfrak{a}^t)$ is independent of c, provided it's nonzero.

Example 3.5. If $R = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, and $\mathfrak{a} = (x_1^{a_1} \ldots x_d^{a_d})$, then we again have

$$\tau(R, \mathfrak{a}^t) = (x_1^{\lfloor a_1 t \rfloor}, \ldots, x_d^{\lfloor a_d t \rfloor}).$$

However, in general the constructions give different things.

Example 3.6. Take $R = k[x, y]_{(x,y)}$, and $\mathfrak{a} = (x^3 + y^2)$. Then in characteristic 0

$$\sup_{t \in \mathbf{R}_{\geq 0}} \{J(R, \mathfrak{a}^t) = R\} = \frac{5}{6},$$

but in characteristic p,

$$\sup_{t\in\mathbf{R}_{\geq 0}} \{\tau(R,\mathfrak{a}^t)=R\} = \begin{cases} 5/6 & p\equiv 1 \mod 6 \\ 5/6-1/(6p) & p\equiv 5 \mod 6 \end{cases}$$

Some basic properties of J and τ (we write (J/τ) to denote "either one"), for $(R,\mathfrak{m})$ regular local:

(1) $(J/\tau)(R,\mathfrak{a}^t)\subseteq(J/\tau)(R,\mathfrak{b}^t)$ if $\mathfrak{a}\subseteq\mathfrak{b}$.

(2) $\mathfrak{a}\subseteq(J/\tau)(R,\mathfrak{a}^1)$.

(3) $(J/\tau)(R,(\mathfrak{q}^{(nh)})^{1/n})\subseteq\mathfrak{q}$ for all $\mathfrak{q}\in\operatorname{Spec}R$ with $h=\operatorname{ht}\mathfrak{q}$.

(4) $(J/\tau)(R,\mathfrak{a}^{s+t})\subseteq(J/\tau)(R,\mathfrak{a}^s)\cdot(J/\tau)(R,\mathfrak{a}^t)$.

(5) $(J/\tau)(R,\mathfrak{a}^{nt})=(J/\tau)(R,(\mathfrak{a}^n)^t)$.

(6) (4) and (5) $\implies (J/\tau)(R,\mathfrak{a}^{nt})\subseteq(J/\tau)(R,\mathfrak{a}^t)^n$.

We have the following deep theorem which relates J to τ by reduction mod p:

Theorem 3.7 (Hara-Takagi-Yoshida [9])**.** *If R is a regular local ring, essentially of finite type over a field k in characteristic p, then we have*

$$J(R,\mathfrak{a}^t)_p=\tau(R_p,(\mathfrak{a}_p)^t)$$

for $p>>0$.

Now we give a proof of Theorem 2.1.

Proof of Theorem 2.1. We basically just combine the properties we listed above.

$$\mathfrak{q}^{(nh)}\subseteq_{(2)}(J/\tau)(R,(\mathfrak{q}^{(nh)})^1)\subseteq_{(5)}(J/\tau)(R,(\mathfrak{q}^{(nh)/n})^n)\subseteq_{(4)}(J/\tau)(R,(\mathfrak{q}^{(nh)})^{1/n})^n\subseteq_{(3)}\mathfrak{q}^n.$$

This completes the proof, and works in characteristic 0 and $p>0$. ☐

So to run the proof in the mixed characteristic case, we need an analog of J or τ satisfying the required properties. The problem is that in mixed characteristic, we don't have a Frobenius and we can't take arbitrary log resolutions.

So we reinterpret J and τ using local duality.

(1) In characteristic 0, let $(R,\mathfrak{m})$ be regular local with $d=\dim R$. It turns out (this needs work) that

$$J(R,\mathfrak{a}^t)=\operatorname{Ann}_R(\{\eta\in H^d_\mathfrak{m}(R):\eta=0 \text{ in } \mathbf{H}^d_\mathfrak{m}(R\pi_*\mathscr{O}_Y(\lfloor t\mathcal{G}\rfloor)\}).$$

This is true by local duality and vanishing of the higher derived images.

(2) In characteristic 0, let $(R,\mathfrak{m})$ be regular local with $d=\dim R$. It turns out (this needs work) that

$$\tau(R,\mathfrak{a}^t)=\operatorname{Ann}_R(\{\eta\in H^d_\mathfrak{m}(R):c^{1/p^e}f^{1/p^e}\eta=0 \text{ in } H^d_\mathfrak{m}(R^{1/p^e} \text{ (or } R^{1/p^\infty})) \text{ for } f\in\mathfrak{a}^{\lceil tp^e\rceil}, e>>0\}).$$

Roughly speaking, these are "elements almost annihilated by $\mathfrak{a}^t$". We use different techniques in both cases to approximate $\mathfrak{a}^t$. So in mixed characteristic, the idea is to replace the perfect algebra R^{1/p^∞} with a perfectoid algebra.

So we want to make sense of $\mathfrak{a}^t$ in $H^d_{\mathfrak{m}}(A)$ for some huge A, in mixed characteristic. This brings us to the work of André in [1]

Theorem 3.8 (André)**.** *Let $(R, \mathfrak{m})$ be a complete regular local ring with $p, x_2, \dots, x_r$ a system of parameters. Then there exists an integral perfectoid R-algebra R_∞ (in the sense of [2]) such that*

(1) All elements of R admit a compatible system of p-power roots, and

(2) $p, x_2, \dots, x_d$ is a (p^{1/p^∞})-almost regular sequence on R_∞.

Now we can define our analog in the perfectoid case, which will be similar to J and τ.

Definition 3.9 (Perfectoid Multiplier/Test Ideal)**.** Now let $(R, \mathfrak{m})$ be a complete regular local ring in mixed characteristic $(0, p)$. We fix R_∞ as in Theorem 3.8, and fix $\mathfrak{a} = (f_1, \dots, f_n) \subseteq R$ and $t \in \mathbf{R}_{\geq 0}$. Then we can define

$$\tau_{R_\infty}(R, \mathfrak{a}^t) = \mathrm{Ann}_R(\{\eta \in H^d_{\mathfrak{m}}(R) : p^{1/p^\infty} f^{1/p^e} \eta = 0 \text{ in } H^d_{\mathfrak{m}}(R_\infty) \text{ for all } f \in \mathfrak{a}^{\lceil (t+\epsilon)p^e \rceil}, \epsilon << 1, \text{ for all } e\})$$

It's natural to consider the additional factor p^{1/p^∞} since R_∞ is almost Cohen-Macaulay. Furthermore, there is an additional $\epsilon << 1$, which we need in the mixed characteristic case, we think.

Remark 3.10. The definition of τ_{R_∞} a priori depends on R_∞, although we don't know the extent to which this is true. But you could start with any integral perfectoid algebra satisfying the conditions in Theorem 3.8, and the construction would work.

Definition 3.11. Let

$$\tau_{R_\infty}(R, [f_1, \dots, f_n]^t) = \mathrm{Ann}_R(S),$$

where

$$S = \{\eta \in H^d_{\mathfrak{m}}(R) : p^{1/p^\infty} f^{1/p^e} \eta = 0 \text{ in } H^d_{\mathfrak{m}}(R_\infty) \text{ for all } f = \prod_i f_i^{a_i}, \sum_i a_i \geq \lceil (t+\epsilon)p^e \rceil\}.$$

The difference here is that we're not considering things like $(f_1 + f_2)^{1/p^e}$. Thus we clearly have

$$\tau_{R_\infty}(R, [f_1, \dots, f_n]^t) \subseteq \tau_{R_\infty}(R, \mathfrak{a}^t).$$

Example 3.12. Let's look at $R = \mathbf{Z}_p[\![x_2, \dots, x_d]\!]$, $\mathfrak{a} = p^{a_1} x_2^{a_2} \dots x_d^{a_d}$. Then

$$\tau_{R_\infty}(R, \mathfrak{a}^t) = p^{\lfloor a_1 t \rfloor} x_1^{\lfloor a_2 t \rfloor} \dots x_d^{\lfloor a_d t \rfloor},$$

so we get the same thing as before.

Remark 3.13. Recently, Shimomoto and André constructed integral perfectoid big Cohen-Macaulay R^+-algebras, although following this approach is much more complicated. For our purposes, R_∞ is enough.

Lastly, we state the main result, involving the perfectoid multiplier/test ideals, as stated in [**?**].

Theorem 3.14 (Ma-Schwede)**.** *τ_{R_∞} satisfies (2), (3) and (5) in the list of properties, and $\tau_{R_\infty}(R, [\underline{f}]^t)$ satisfies (2)-(5).*

Thus we can run essentially the same proof as in Theorem 2.1. The only subtlety is that we need to choose a generating set compatible with all of the previous steps, but this is easy to account for.

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