BIG CM MODULES, MORPHISMS OF PERFECT COMPLEXES, INTERSECTION THEOREMS

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1. Disclaimer

This is the transcription of a lecture given by Luchezar Avramov for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture concerns new derivations of some of the classical consequences of the existence of big Cohen-Macaulay modules, and new relations between the theorems involved in the homological conjectures. This is Joint work with Srikanth Iyengar and Amnon Neeman.

2. Introduction

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with unity. The Improved New Intersection Conjecture (INIT), proven in [3], says

Theorem 2.1. *Given a complex of finite free R-modules*

$$\mathcal{F} = 0 \to F_d \to \cdots \to F_0 \to 0$$

such that

(1) $\mathfrak{m}^n H_i(F) = 0$ *for* $i > 0$*, and*

(2) There exists $z \in H_0(\mathcal{F}) \setminus \mathfrak{m} H_0(\mathcal{F})$ *such that* $\mathfrak{m}^n z = 0$.

Then $d \geq \dim R$.

The original proof in [3], is done using big Cohen-Macaulay modules. Hochster, in [4] showed that it also follows from the Canonical Element Conjecture (CEC). Dutta in t[2] showed that in fact, CEC $\iff$ INIT.

Remark 2.2. We actually have $d = \dim R$ for the Koszul complex $K(\mathbf{x})$ where $\mathbf{x}$ is a system of parameters for R.

We want to show that the statement of the theorem can be improved. In particular, we will change the length d of the complex as a parameter in the theorem.

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3. Perfect Complexes

Definition 3.1. Let R be a ring and $\mathcal{F}$ be a bounded complex of R-modules. We define $\text{level}^R(\mathcal{F})$ to be the minimal number of extensions which are necessary to build $\mathcal{F}$ from complexes of finitely generated projective modules with zero differential maps.

This is motivated from Beilinson-Bernstein-Deligne's way of building thick subcategories of derived categories.

Proposition 3.2.

$$\text{level}^R(\mathcal{F}) = \inf\{n \in \mathbf{N} : F \simeq Q \text{ with a semi-projective filtration}\},$$

where $F \simeq Q$ is taken in the derived category, and Q is a direct summand of a complex P with a semi-projective filtration, i.e. a sequence

$$0 = P_0 \subseteq P_1 \subseteq \ldots \subseteq P_\ell = P$$

such that P_i/P_{i-1} is a complex of finitely generated projective modules with 0 differentials.

Now in the INIT, our new statement, following [1], is

Theorem 3.3.

$$\text{level}^R(\mathcal{F}) - 1 \geq \dim R.$$

Example 3.4.

(1) If $\mathcal{F}$ is a minimal projective resolution of a finitely generated R-module M, then $\text{level}^R(\mathcal{F}) = \text{pd}(M) + 1$.

(2) $\text{level}^R(\mathcal{F}) \leq \text{span}(\mathcal{F}) + 1$.

(3) $\text{span}(\mathcal{F}) - \text{level}^R(\mathcal{F})$ can be arbitrarily large.

Definition 3.5. A ghost map in $D(R)$ is a map $f : X \to Y$ with $H_i(f) = 0$ for all i.

Definition 3.6. A complex $X \in D(R)$ is I-torsion for some ideal $I \subseteq R$ if every element of $H(X)$ is annihilated by some power of I.

Remark 3.7. There exists a map of complexes $t : R\Gamma_I(X) \to X$ for every X where $R\Gamma_I(X)$ is an I-torsion complex and t universal for such a map.

Proposition 3.8. *Let C be a balanced big Cohen-Macaulay R-module for a local Noetherian R. Then*

(1) $t : R\Gamma_I C \to C$ is a composition of c ghosts, where c is the height of the the ideal I, for any ideal $I \subseteq R$.

(2) (Ghost Lemma) If $g : G \to C$ factors through $G \to X \to C$ with X an I-torsion complex and $\text{level}^R(G) \leq c$, then $g = 0$ in $D(R)$.

Proof. We will prove (1). We can assume $I = (x_1, \ldots, x_c)$ where $x_1, \ldots, x_c$ is part of a system of parameters for R. We have

$$R\Gamma_{x_1,\ldots,x_c}(C) \to R\Gamma_{x_1,\ldots,x_{c-1}}(C) \to \cdots \to R\Gamma_{x_1}(C) \to C.$$

Since the x_i are part of a system of parameters, $H(R\Gamma_{x_1,\ldots,x_i})$ lives in degree i. Then clearly the maps are ghosts. ☐

Definition 3.9. A map $\mathcal{G} \to \mathcal{F}$ of complexes of R-modules is called tensor-nilpotent if there exists an n such that $f^{\otimes^{\mathbf{L}} n} : \mathcal{G}^{\otimes^{\mathbf{L}} n} \to \mathcal{G}^{\otimes^{\mathbf{L}} n}$ is zero.

We will need the following result from homotopy theory, proved in [5] and [6].

Theorem 3.10 (Hopkins, Neeman)**.** *If $\mathcal{F}$ and $\mathcal{G}$ are perfect complexes of R-modules, then tensor nilpotence of $f : \mathcal{G} \to \mathcal{F}$ is the same as being fiberwise-zero, i.e.*

$$\kappa(\mathfrak{p}) \otimes f : \kappa(\mathfrak{p}) \otimes_R G \to \kappa(\mathfrak{p}) \otimes_R \mathcal{F}$$

is a ghost map for all $\mathfrak{p} \in \operatorname{Spec} R$.

Theorem 3.11. *Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism of perfect complexes of R-modules and let $I \subseteq R$ be an ideal. Then if*

(1) f factors through a complex X with $H(X)$ I-torsion and

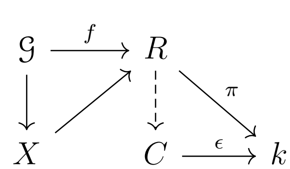
(2) $\operatorname{level}^R(\operatorname{Hom}(\mathcal{G}, \mathcal{F})) \le \operatorname{ht} I$,

then f is fiberwise zero, hence tensor nilpotent.

Proof Sketch. First reduce to the case when $\mathcal{F} = R$, by using basic properties of level^R. Next, reduce to the case where $R = (R, \mathfrak{m}, k)$ is local and $\mathfrak{p} = \mathfrak{m}$, and assume that I is $\mathfrak{m}$-primary. Then we have to deal with

$$\mathcal{G} \xrightarrow{f} R,$$

which factors through some X as $\mathcal{G} \to X \to R$. Note we have a big Cohen-Macaulay module C with a surjection $C \to k$. So lift the map $R \to k$ to a map $R \to C \to k$. This now gives a diagram



By Proposition 3.8, the map $G \to X \to R \to C$ is zero. There is a map

$$H(G \otimes k) \to H(R \otimes k) \to H(k \otimes k)$$

This is 0, and $H(R \otimes k) \to H(k \otimes k)$ is an isomorphism, thus $H(f \otimes k) = 0$. ☐

4. Applications

We have the first following interpretation.

Theorem 4.1 (Morphic Intersection Theorem)**.** *Suppose R is Noetherian and $f : \mathcal{G} \to \mathcal{F}$ is a morphism of perfect complexes. If f is not fiberwise zero, and factors through some X with I-torsion homology for some I, then*

$$\operatorname{span}(\mathcal{F}) + \operatorname{span}(\mathcal{G}) \geq \operatorname{level}^R(\operatorname{Hom}_R(\mathcal{G}, \mathcal{F})) \geq \operatorname{ht} I + 1.$$

The first part is an easy lemma, the second inequality follows from Theorem 3.11. Next, we have

Theorem 4.2 (Canonical Element Conjecture)**.** *Let R be a local Noetherian ring, and $\underline{x}$ be a system of parameters, with $\mathcal{F}$ a perfect complex of R-modules, with $f : K(\underline{x}) \to \mathcal{F}$ has $H_0(k \otimes f) \neq 0$, where K is the Koszul complex.*

Then $H_{\dim R}(S \otimes f) \neq 0$ with $S = R/\underline{x}$.

As a corollary, we obtain

Corollary 4.3. *If $\underline{x}$ is as before and $\underline{x} \subseteq (\underline{y})$ for some $\underline{y}$, so for example $A\underline{y} = \underline{x}$, then the $d \times d$ minor ideal*

$$I_d(A) \nsubseteq (\underline{x}).$$

For example, if we take $x_i = y_i^n$ where $y_1, \ldots, y_d$ is a system of parameters for R, then this corollary recovers the monomial conjecture in the classical form.

References

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