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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL **Email/Phone:** kkoziol@ualberta.ca

Speaker's Name: XINWEN ZHU

Talk Title: OVERVIEW: RECENT PROGRESS OF LANGLANDS CORRESPONDENCE OVER FUNCTION FIELDS

Date: 4 / 8 / 19 **Time:** 9 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: THE SPEAKER GAVE AN INTRODUCTION TO THE (CLASSICAL) LANGLANDS PROGRAM, AND THEN EXPLAINED THE FUNCTION FIELD ANALOG, HIGHLIGHTING RESULTS OF V. LAFFORGUE.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Recent progress of Langlands correspondence over function fields

Xinwen Zhu

California Institute of Technology

MSRI
April 8, 2019

Overview

Set-up

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- $\mathcal{A}_G = \varinjlim_K C(G(F)\backslash G(\mathbb{A})/K, \mathbb{C})$, on which the Hecke algebra $\mathbb{T}_K = C_c(K\backslash G(\mathbb{A}^\infty)/K, \mathbb{C})$ acts by convolution, where K runs over open compact subgroups of $G(\mathbb{A}^\infty)$.

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- ${}^L G = \hat{G} \rtimes \mathrm{Gal}(\widetilde{F}/F)$ the L-group, where $\widetilde{F}$ is the splitting field of G.
- ${}^L G_v$ local L-group. For almost all places v, ${}^L G_v := \hat{G} \rtimes \langle \mathrm{Frob}_v \rangle$.

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There is a natural bijection between two sets

$$\{\text{Automorphic representations } \pi \text{ of } G\} \\ \leftrightarrow \{\text{Langlands parameters } \phi : \mathcal{L}_F \to {}^L G \text{ up to } \hat{G}\text{-conjugacy}\},$$

matching the spectral data arising from the action of $\mathbb{T}_K$ on π^K, and the arithmetic data arising from ϕ.

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(2) Every automorphic representation decomposes as restricted tensor product

$$\pi = \bigotimes_v{}' \pi_v,$$

with each π_v an irreducible representation of $G(F_v)$, and for almost all v, $\dim \pi_v^{G(\mathcal{O}_v)} = 1$;

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(3) Satake isomorphism (depending on the choice $\sqrt{\sharp k_v} \in \mathbb{C}$)

$$\mathrm{Sat}_v : \mathbb{T}_v := C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v)) \cong \mathbb{C}[\hat{G}\,\mathrm{Frob}_v]^{\hat{G}},$$

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From (1)-(3), we see that for an irreducible automorphic representation π, and for almost all v, $\mathbb{T}_v$ acts on $\pi_v^{G(\mathcal{O}_v)}$ by a character, giving a semisimple conjugacy class $c(\pi_v) \subset {}^LG_v$.

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The collection $\{c(\pi_v)\}_v$ is called the Hecke eigensystem attached to π.

Automorphic Langlands group

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If F is a function field, $\mathcal{L}_F$ should be equal to $\mathcal{G}_F$, which was constructed unconditionally by Drinfeld. (But the construction uses the Langlands correspondence.)

Galois representation

The set of homomorphisms $\phi : \mathcal{G}_F \to {}^L G$ up to conjugacy can be replaced by (depending on a choice $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$)

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A general $\phi : \mathcal{L}_F \to {}^L G$ gives $\{\phi_v(\mathrm{Frob}_v)\}$ similarly. (For almost all v, the localized parameter $\phi_v : \mathcal{L}_v \to {}^L G_v$ is unramified.)

Langlands correspondence (naive form)

The natural bijection

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The bijection should be compatible at *all* local places once the local Langlands correspondence is introduced.

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On the contrary, over number fields, although a lot of progresses have been made, there are still outstanding questions for $\mathrm{GL}_2/\mathbb{Q}$.

For the group other than GL_n, the situation is far more complicated.

Element conjugacy v.s. global conjugacy

First, there exists *different* π and π' with the same Hecke eigensystem, and the Langlands correspondence will not be a bijection.

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Over function fields, V. Lafforgue's introduced the excursion algebra as an enlargement of the (unramified) Hecke algebra, which overcomes this difficulty.

Cohomology of modular varieties

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- $\pi \mapsto W(\pi)$ realizes part of the Langlands correspondence for $\mathrm{GL}_2/\mathbb{Q}$

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For example, the multiplicity space $W(\pi)$ usually is not the "native" Galois representation one would expect. Even it is, it does not determine the Langlands parameter associated to π.

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- ${}^L G = \hat{G}$ is defined over E.

Lafforgue's theorem: Statement

Theorem (V. Lafforgue)

- *For every finite set I, there is an E-linear functor*

$$H_I^{\text{cusp}} : \text{Rep}(\hat{G}^I) \to \mathbb{T}_K[\Gamma_F^I]\text{--Mod};$$

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- *For every $\phi : I \to J$, there is a canonical isomorphism of functors*

$$\chi_\phi : \mathrm{Res}_\phi \circ H_I^{\mathrm{cusp}} \cong H_J^{\mathrm{cusp}} \circ \mathrm{Res}_\phi,$$

where Res_ϕ in both sides denote the natural restriction functors induced by $\hat{G}^J \to \hat{G}^I$ and $\Gamma_F^J \to \Gamma_F^I$.

Theorem (Cont'd)

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- $H_{\emptyset}^{\text{cusp}}(\mathbf{1}) = C_{\text{cusp}}(G(F)\Xi\backslash G(\mathbb{A})/K)$, *where* $\mathbf{1}$ *denotes the trivial representation;*
- *The representation* Γ_F^I *on* $H_I^{\text{cusp}}(W)$ *is continuous and unramified almost everywhere;*

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The above two data satisfy the following conditions

- $H_{\emptyset}^{\text{cusp}}(\mathbf{1}) = C_{\text{cusp}}(G(F)\Xi\backslash G(\mathbb{A})/K)$, *where* $\mathbf{1}$ *denotes the trivial representation;*
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Roughly speaking $H_I^{\mathrm{cusp}}(W)$ is the (cuspidal) cohomology of moduli of Shtukas associated to $W \in \mathrm{Rep}(\hat{G}^I)$.

Drinfeld $\mathcal{O}$-module

Given the system of functors $\{H_I\}$ as in the above theorem, Lafforgue constructed the action of a large commutative algebra (so-called excursion algebra) on each $H_I(W)$.

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We give a more conceptual explanation of Lafforgue's construction (following Drinfeld's idea).

First, if Γ is an abstract group, the set of homomorphisms from Γ to $\hat{G}$ is represented by an affine E-scheme $\mathrm{Hom}(\Gamma, \hat{G})$, on which $\hat{G}$ acts by conjugation.

Proposition

Given a system of functors

$$\{H_I : \mathrm{Rep}(\hat{G}^I) \to \mathrm{Rep}(\Gamma^I)\}$$

satisfying similar conditions as in Lafforgue's theorem, let $\mathfrak{A} = H_{\{0\}}(\mathrm{Reg})$, *where* $\mathrm{Reg} = E[\hat{G}]$ *denotes the regular representation of* $\hat{G}$. *Then*

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- *There is a natural isomorphism* $H_I(W) \cong (\mathfrak{A} \otimes W)^{\hat{G}}$ *of* Γ^I*-modules.*

Pseudo-representations

It follows that every $H_I(W)$ is acted by the (abstract) excursion algebra

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Theorem (V. Lafforgue)

The natural map from the set of semisimple representations $\sigma : \Gamma \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ up to $\hat{G}$-conjugacy to the set of $\hat{G}$-valued pseudo-representations is a bijection.

Frobenius-Hecke compatibility

Let v be an unramified prime. The Frobenius conjugacy class at v defines a canonical map $\mathrm{Hom}(\Gamma_F, \hat{G})/\hat{G} \to \hat{G}/\hat{G}$, and therefore induces a canonical map

$$E[\hat{G}]^{\hat{G}} \to E[\mathrm{Hom}(\Gamma_F, \hat{G})]^{\hat{G}} = \mathcal{B}.$$

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The Frobenius-Hecke compatibility (which we refer as $S = T$ theorem) says that the induced action of $E[\hat{G}]^{\hat{G}}$ on $H_I^{\mathrm{cusp}}(W)$ coincides with the action of the unramified Hecke algebra $\mathbb{T}_v$ under the Satake isomorphism.

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Definition

A Langlands parameter $\phi : \Gamma_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ is called elliptic if $\mathfrak{S}_\phi$ is finite.

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Theorem (V. Lafforgue-Z.)

1. *For each elliptic Langlands parameter $\phi : \Gamma_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, the fiber $\mathfrak{A}_\phi$ of $\mathfrak{A}$ at ϕ is a finite dimensional $\mathbb{T}_K[\mathfrak{S}_\phi]$-module.*

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2. *Let $\chi : \mathcal{B} \to \overline{\mathbb{Q}}_\ell$ be the character corresponding to ϕ and $H_I^{\mathrm{cusp}}(W)_\chi$ the localization of the $\mathcal{B}$-module $H_I^{\mathrm{cusp}}(W)$ at the maximal ideal of the kernel of χ. Then*

$$H_I^{\mathrm{cusp}}(W)_\chi = (\mathfrak{A}_\phi \otimes W_{\phi^I})^{\mathfrak{S}_\phi}$$

Local Langlands category

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As we learned from the global theory, we shall enlarge this category (which in some sense would be the local counterpart of $H_\emptyset(\mathbf{1})$).

It turns out that the more fundamental object here is the category of sheaves on $B(G)$ (the quotient of $G(L)$ by the Frobenius conjugation, where L is the completion of the maximal unramified extension of F_v.)

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Whatever this category is, $[1/G(F_v)] \subset B(G)$ and therefore this category should contain the category of smooth representations of $G(F_v)$ as a full subcategory.

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Another approach is via the category of Frobenius conjugate equivariant sheaves on the loop group of G.

The loop group LG of G is the affine (perfect) ind-scheme over k_v which represents the functor

$$\mathrm{Alg}_{k_v} \to \mathrm{Grp}, \quad R \mapsto G(W_{\mathcal{O}_v}(R)[1/\varpi_v]),$$

where $W_{\mathcal{O}_v}(-)$ is the Witt vector with coefficients in $\mathcal{O}_v$.

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In particular, there will be objects δ_{reg_n} in this category whose endomorphism is the (opposite) of the Hecke algebra $C_c(K_n\backslash G(F_v)/K_n)$, where K_n is the level n-congruence subgroup of G.

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Theorem (Genestier-Lafforgue)

1. *For every $\Lambda_m := \mathcal{O}_E/\varpi_E^m$, and every n, there is a canonical map*

$$\mathcal{B}_{v,\Lambda_m} \to \mathcal{Z}(\mathrm{End}\delta_{\mathrm{reg}_n,\Lambda_m}),$$

where $\mathcal{B}_{v,\Lambda_m}$ is the local excursion algebra with Λ_m-coefficient, and $\delta_{\mathrm{reg}_n,\Lambda_m}$ denotes the sheaf with Λ_m-coefficient.

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3. *The resulting map is compatible with the global Langlands parameterization.*