MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Karol Koziol Email/Phone: kkoziol@ualberta.ca

Speaker's Name: Charlotte Chan

Talk Title: Drinfeld's Lemma

Date: 4/10/19 Time: 9:00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The speaker discussed Drinfeld's lemma, namely that the action of a certain geometric Galois group of a product factors through the product

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Drinfeld's Lemma — Chan

- X sm. projective curve $/\mathbb{F}_q$
- $F = \mathbb{F}_q(X)$
- $F_I = \mathbb{F}_q(X^I)$
- $\eta \in X$ generic pt, $\bar{\eta}$ geom generic pt
- $\eta_I \in X^I$ ——— " ———, $\bar{\eta}_I$ ——— " ———
- $X \hookrightarrow X^I$
- $\eta \longmapsto \Delta\eta$
- $E / \mathbb{Q}_\ell$ finite

Have speciali'n map

$$sp : \bar{\eta}_I \longrightarrow \Delta\bar{\eta}$$

morphism of schemes

$$\bar{\eta}_I \longrightarrow X^I_{(\Delta\bar{\eta})} \longrightarrow \mathrm{Spec}\left(\bar{F} \otimes_{\bar{\mathbb{F}}_q} \cdots \otimes_{\bar{\mathbb{F}}_q} \bar{F}\right)$$

So in particular

$$\bar{F} \otimes_{\bar{\mathbb{F}}_q} \cdots \otimes_{\bar{\mathbb{F}}_q} \bar{F} \hookrightarrow \bar{F}_I$$

$$\rightsquigarrow \mathrm{FWeil}(\eta_I, \bar{\eta}_I) = \left\{ \delta \in \mathrm{Aut}_{\bar{\mathbb{F}}_q}(\bar{F}_I) : \delta\big|_{F_I^{\mathrm{perf}}} = \prod_{i=1}^{|I|} F_{\{i\}}^{n_i},\ n_i \in \mathbb{Z} \right\}$$

$$\begin{pmatrix}\text{depends} \\ \text{on} \\ sp\end{pmatrix} \quad \mathrm{FWeil}(\eta_I, \bar{\eta}_I) \downarrow \mathrm{Weil}(\eta, \bar{\eta})^I, \qquad \delta \mapsto \left(\left(\delta \circ \mathrm{Frob}^{-n_i}\right)\big|_{i^{\text{th}} \text{ comp of } \bar{F}} \right)_{i \in I}$$

REMINDER

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\mathrm{GEOM}}(\eta_I, \bar{\eta}_I) \arrow[r] \arrow[d, two heads] & \mathrm{FWGL}(\eta_I, \bar{\eta}_I) \arrow[r] \arrow[d, two heads] & \mathbb{Z}^I \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \pi_1^{\mathrm{GEOM}}(\eta, \bar{\eta})^I \arrow[r] & \mathrm{WGL}(\eta, \bar{\eta})^I \arrow[r] & \mathbb{Z}^I \arrow[r] & 0
\end{tikzcd}$$

THM (DRINFELD'S LEMMA) LET $Q = \mathrm{FWEIL}(\eta_I, \bar{\eta}_I) / \bigcap U$

$U \subset \pi_1^{GEOM}(\eta_I, \bar{\eta}_I)$ OPEN

$U \lhd \mathrm{FWEIL}(\eta_I, \bar{\eta}_I)$

1) EVERY F.D. CTS REP V OVER E OF $\mathrm{FWEIL}(\eta_I, \bar{\eta}_I)$ DESCENDS TO A REP OF Q

2) CHOICE OF SP INDUCES AN ISOM $Q \cong \mathrm{WEIL}(\eta, \bar{\eta})^I$

(SEE REMINDER)

PF OF 1

$\rho : \mathrm{FWEIL}(\eta_I, \bar{\eta}_I) \longrightarrow GL_n(E)$ CTS

- $\mathrm{KER}\left(\rho\big|_{\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)}\right) \lhd \mathrm{FWEIL}(\eta_I, \bar{\eta}_I)$
- NEED $\underbrace{\mathrm{KER}\left(\rho\big|_{\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)}\right)}$ IS ~~OPEN IN $\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)$~~ CONTAINS $\bigcap U$

ONLY KNOW $\mathrm{KER}(\cdots)$ IS CLOSED

$\mathrm{KER}(\cdots) = H$

- CLAIM $\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)/H$ IS TOP F.G.

PF $\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)/H \xrightarrow{\sim} \underbrace{\rho\left(\pi_1^{GEOM}(\eta_I, \bar{\eta}_I)\right)}_{\text{CPCT}} \subset GL_n(E)$

AND COMPACT SUBGPS OF $GL_n(E)$ ARE TOP F.G. ✓

- $\underbrace{\pi_1^{\text{GEOM}}(\eta_I, \bar\eta_I)/H}$ PROFINITE

$\Rightarrow$

$$\cong \varprojlim_{\substack{A \\ \text{FINITE} \\ \text{GP}}} \pi_1^{\text{GEOM}}(\eta_I, \bar\eta_I)/U_A$$

WHERE $U_A = \bigcap_{\substack{f \text{ CTS}, \\ \text{TRIV ON} \\ H}} \ker\left(\pi_1^{\text{GEOM}}(\eta_I, \bar\eta_I) \xrightarrow{f} A\right)$

ONLY FINITELY MANY ~~MAPS~~ f's IN THIS INTERSECTION $\Rightarrow$ U_A IS OPEN

$\Rightarrow H \supset \bigcap_{\dots} U$

IN FACT,

<u>COR</u> A F.G COMM RING $/E$, M AN A-MOD OF FINITE TYPE, THEN ANY A-LINEAR ACTION OF $\text{FWEIL}(\eta_I, \bar\eta_I)$ ON M FACTORS THROUGH Q

"CTS" = ~~$M = \underbrace{\bigcup \text{F.D}_{/E} \text{ SUBMODS}}$~~
~~$\pi_1^{\text{GEOM}}(\eta_I, \bar\eta_I)$ STABLE~~ w/
~~CTS ACTION~~

$= \text{FWEIL}(\eta_I, \bar\eta_I) \circlearrowright M/\mathfrak{m}^n M$ IS CTS $\forall\, \mathfrak{m} \subset A$ MAXL $\forall\, n \geq 1$

PF OF COR $\mathfrak{m} \subset A$ MAX'L IDEAL

$\Rightarrow A/\mathfrak{m}$ IS F.D. OVER E

$\Rightarrow M/\mathfrak{m}^n M$ IS F.D. OVER E $\forall n$

$\Rightarrow M/\mathfrak{m}^n M \circlearrowleft Q$

B/C $\delta \in \bigcap_{\dots} U$, $x \in M$

$$\Rightarrow \underbrace{\delta x}_{-x} \in \bigcap_{\mathfrak{m} \subset A} \bigcap_{n \geq 1} \mathfrak{m}^n M = 0 \qquad \square$$

PF OF 2) WANT TO SHOW

$$\circledast \quad \bigcap_{\dots} U = \mathrm{KER}\left(\mathrm{FWEIL}(\eta_I, \bar{\eta}_I) \longrightarrow \mathrm{WEIL}(\eta, \bar{\eta})^I \right)$$

SANITY CHECK: $I = \{1\}$

$$\bigcap_{\dots} U \subset \bigcap_{\substack{Z_0 \to X \\ \text{COVER}}} \left(\underbrace{\pi_1^{\text{GEOM}}(\eta_I, \bar{\eta}_I) \cap \mathrm{GAL}(\bar{F}/\mathbb{F}_q(Z_0))}_{\text{SUBSET OF } U\text{'S APPEARING ON LHS}} \right)$$

$$= \{1\}$$

SUPPOSE FIRST $I = \{1, 2\}$

$\circledast \subset$

$$\delta \in \bigcap_{\dots} U \subset \bigcap_{Z_0 \to X} \mathrm{GAL}\left(\bar{F}_I / \mathbb{F}_q(Z_0^I)\right) \cap \pi_1^{\text{GEOM}}(\eta_I, \bar{\eta}_I)$$

$$\Rightarrow \delta_{*}\big|_{\mathbb{F}_q(Z_0^I)} = 1 \quad \forall \text{ COVERS } Z_0 \longrightarrow X \Rightarrow \delta \in \mathrm{KER}$$

$(*)$ $\supset$ WILL PROVE $U \supset \mathrm{KER}(\cdots)$

CONSIDER $F' := \mathbb{F}_q(X \times \bar{\eta})$

$$F_I \subset F' \subset \overline{F_I}$$

$$U \cap \underbrace{\mathrm{GAL}\left(\overline{F_I}/\mathbb{F}_q(X \times \bar{\eta})\right)}_{\mathrm{GAL}(\overline{F_I}/F') \,\cap\, \pi_1^{\mathrm{GEOM}}(\eta_I, \bar{\eta}_I)}$$

$\longleftrightarrow$ FINITE EXT'N OF F'

$\longleftrightarrow$ COVER $Z \longrightarrow X \times \bar{\eta}$

SO $U \cap \mathrm{GAL}\left(\overline{F_I}/\mathbb{F}_q(X \times \bar{\eta})\right) = \mathrm{GAL}\left(\overline{F_I}/\mathbb{F}_q(Z)\right)$

SUPPOSE WE KNEW $Z = Z_0 \times \bar{\eta}$, WHERE $Z_0 \longrightarrow X$ IS A COVER

THEN $\delta \in \mathrm{KER}$, $\delta : \overline{F_I} \longrightarrow \overline{F_I}$ FIXING $F \otimes_{\mathbb{F}_q} \bar{F}$

$\Longrightarrow \delta$ FIXES $\mathbb{F}_q(Z_0) \otimes \bar{F}$

$\Longrightarrow \delta \in \mathrm{GAL}\left(\overline{F_I}/\mathbb{F}_q(Z)\right) \subset U$

$$\boxed{\begin{array}{l}
\text{LEMMA (DRINFELD)} \quad Y \text{ PROJ SCH}/\overline{\mathbb{F}_q}, \quad L \text{ ALG CLOSED FIELD } L \supset \mathbb{F}_q \\
\left\{\begin{array}{c}\text{COH SHEAVES}\\ \text{OF } \mathcal{O}_Y\text{-MODS}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{COH SHEAVES}\\ \mathcal{A} \text{ OF } \mathcal{O}_{Y \otimes L}\text{-MODS W/}\\ (\mathrm{id}_Y \times \mathrm{FR}_L)^* \mathcal{A} \xrightarrow{\sim} \mathcal{A}\end{array}\right\} \\
\qquad \mathcal{F} \longmapsto \mathrm{pr}_Y^* \mathcal{F}
\end{array}}$$

6

IF $Z \longrightarrow X \times \bar{\eta}$ IS FÉT COVER, THEN

$$U \triangleleft \mathrm{FWEIL}(\eta_I, \bar{\eta}_I) \Longrightarrow id_X \times FR_{\mathbb{F}} \in \mathrm{AUT}(X \times \bar{\eta})$$

LIFTS TO AN AUT OF Z

LEMMA $\Longrightarrow$ $Z = Z_0 \times \bar{\eta}$ ✓

IF $Z \longrightarrow X \times \bar{\eta}$ IS NOT FÉT, NEED MORE

CAN PROVE:

IF U QPROJ NORMAL $/\mathbb{F}_q$, THEN

$$\{ \text{FÉT } U\text{-SCH} \} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{FÉT } U_L \text{ SCHEME EQUIPPED} \\ \text{W/ FROB}_L \end{array} \right\}$$

LOCAL VERSION:

$$L = \mathbb{F}_q((t_1))((t_2)) \cdots ((t_n))$$

THEN
$$\begin{array}{c} \mathrm{FWEIL}(L, \bar{L}) \\ \downarrow \\ \mathrm{WEIL}(\mathbb{F}_q((t)), \overline{\mathbb{F}_q((t))})^I \end{array}$$

$$\boxed{\begin{array}{l} \underline{\text{LEMMA}} \text{ (GENESTIER - LAFFORGUE)} \\ \left\{ \begin{array}{c} \text{ÉT ALG OVER } L \\ \text{W/ PARTIAL FROB} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{FINITE SETS W/ CTS ACTION} \\ \prod_{i=1}^{n} \mathrm{GAL}\left(\overline{\mathbb{F}_q((t_i))}/\mathbb{F}_q((t_i))\right) \end{array} \right\} \end{array}}$$