MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Karol Kozlol Email/Phone: kkozlol@ualberta.ca

Speaker's Name: Jack Thorne

Talk Title: Potential Automorphy of $\hat{G}$-local systems

Date: 4 / 12 / 19 Time: 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The speaker discussed his joint work with Böckle - Harris - Khare which proves potential automorphy of certain Galois parameters of function fields. This can be viewed as a partial converse to results of V. Lafforgue.

CHECK LIST

(This is **NOT** optional, we will **not pay for incomplete forms**)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

POTENTIAL AUTOMORPHY OF $\hat{G}$-LOCAL SYSTEMS

— THORNE

"PARTIAL CONVERSE" TO V. LAFFORGUE'S WORK

NOTATION

- X SMOOTH PROJ GEOM CONNECTED CURVE $/ \mathbb{F}_q$
- $K = \mathbb{F}_q(X)$
- v PLACE OF $K \rightsquigarrow K_v, \mathcal{O}_{K_v}, \varpi_v, k(v) = \mathcal{O}_{K_v}/\varpi_v$
- $\mathbb{A}_K = \prod_v{}' K_v \supset \widehat{\mathcal{O}}_K = \prod_v \mathcal{O}_{K_v}$
- $\Gamma_K = \pi_1^{\text{ét}}(X) \supset \overline{\Gamma}_K = \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q})$

  EVERYWHERE UNRAMIFIED

- $G / \mathbb{F}_q$ CONNECTED ~~REDUCTIVE~~, SPLIT, SEMISIMPLE
- $\hat{G} / \mathbb{Z}$ DUAL GP (SPLIT)
- $\ell \nmid q$ PRIME
- $\mathcal{A}_K = C_{cusp}(G(K) \backslash G(\mathbb{A}) / G(\widehat{\mathcal{O}}_K), \overline{\mathbb{Q}}_\ell)$

  FD. $\overline{\mathbb{Q}}_\ell$-V.S.

WE HAVE COMMUTATIVE $\overline{\mathbb{Q}}_\ell$-SUBALGS OF $\mathrm{END}_{\overline{\mathbb{Q}}_\ell}(\mathcal{A}_K)$

↳ $\mathbb{T}_K = \overline{\mathbb{Q}}_\ell$-ALG GEN'D BY $T_{v,f}$, V PLACE OF K,
$f \in \mathbb{Z}[\hat{G}]^{\hat{G}}$

↳ $B_K = \overline{\mathbb{Q}}_\ell$-ALG GEN'D BY $S_{n,\underline{\gamma},f}$ EXCURSION OP'S
$(n \in \mathbb{N},\ \underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in \Gamma_K^n,\ f \in \mathbb{Z}[\hat{G}^n]^{\hat{G}})$

WE KNOW $S_{1, \mathrm{FROB}_v, f} = T_{v,f} \Rightarrow \mathbb{T}_K \subset B_K$

THM (V. LAFFORGUE) FOR ANY $\alpha : B_K \longrightarrow \overline{\mathbb{Q}}_\ell$,
$\exists$ A CTS $\hat{G}$-CR HOMOMORPHISM ("CR" = "COMPLETELY REDUCIBLE" ⇝ SEMI SIMPLE)
$$\sigma_\alpha : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$$
ST
$\forall\ n, \underline{\gamma}, f,$
$$f(\sigma_\alpha(\gamma_1), \dots, \sigma_\alpha(\gamma_n)) = \alpha(S_{n,\underline{\gamma},f})$$
MOREOVER, THIS DETERMINES σ_α UP TO $\hat{G}(\overline{\mathbb{Q}}_\ell)$-CONJ

DEF LET $\sigma : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ BE A CTS $\hat{G}$-CR REP
WE SAY σ IS CUSPIDAL AUTOMORPHIC IF $\exists$
$\alpha : B_K \longrightarrow \overline{\mathbb{Q}}_\ell$ ST $\sigma \sim \sigma_\alpha$ ($\hat{G}$-CONJ)

THM 1 (BÖCKLE - HARRIS - KHARE - THORNE)

SUPPOSE $\sigma : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ CTS w/ ZARISKI-DENSE IMAGE, THEN $\exists$ A FINITE GALOIS EXT L/K ST $\sigma|_{\Gamma_L}$ IS CUSPIDAL AUTOMORPHIC

SHOULD BE ABLE TO REMOVE "UNRAMIFIED" AND "SEMISIMPLE G", "ZARISKI-DENSE" IS MORE SUBTLE

NOTE IF $\sigma, \sigma' : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ ARE TWO HOMS ST

1) σ HAS ZARISKI-DENSE IMAGE
2) $\forall f \in \mathbb{Z}[\hat{G}]^{\hat{G}}$, $\forall v$, $f(\sigma(\mathrm{FROB}_v)) = f(\sigma'(\mathrm{FROB}_v))$

THEN $\sigma \sim \sigma'$.

THIS MEANS COMPATIBLE SYSTEMS OF DENSE IMAGE REPS ARE WELL-BEHAVED

§ DEFORMATION THEORY

$E/\mathbb{Q}_\ell$ FINITE EXT, $E \supset \mathcal{O} \ni \varpi$, $k = \mathcal{O}/\varpi$

$\mathcal{C}_\mathcal{O}$ = COMPLETE NOETHERIAN $\mathcal{O}$-ALGS, LOCAL, w/ RES. FLD k

FIX $\bar{\sigma} : \Gamma_K \longrightarrow \hat{G}(k)$

WE DEFINE FUNCTORS:

- $\mathrm{LIFT}_{\bar{\sigma}} : \mathcal{C}_\mathcal{O} \longrightarrow \mathrm{SETS}$
  $A \longmapsto \{\sigma : \Gamma_K \to \hat{G}(A) : \sigma \bmod \mathfrak{m}_A = \bar{\sigma}\}$

- $\mathrm{DEF}_{\bar\sigma} : \mathcal{C}_O \longrightarrow \text{SETS}$

$$A \longmapsto \mathrm{LIFT}_{\bar\sigma}(A) / \mathrm{KER}(\hat G(A) \to \hat G(k))$$

LET $\bar t = \mathrm{TR}(\bar\sigma)$ ASS'D PSEUDO CHAR IE $\bar t = (\bar t_m)_{m \geq 1}$

$$\bar t_m : \Gamma_K^m \longrightarrow (\hat G^m / \hat G)(k) \quad + \text{ AXIOMS}$$

- $\mathrm{PDEF}_{\bar\sigma} : \mathcal{C}_O \longrightarrow \text{SETS}$

$$A \longmapsto \{ t : \Gamma_K \to A : t \bmod \mathfrak{m}_A = \bar t \}$$

THERE'S A NATURAL TRANSFORMATION

$$\mathrm{DEF}_{\bar\sigma} \longrightarrow \mathrm{PDEF}_{\bar\sigma}$$
$$\sigma \longmapsto \mathrm{TR}(\sigma)$$

THM 2 LET $\ell \nmid |W|$ (WEYL GP), $\bar\sigma$ ABS $\hat G$-IRR, AND $Z_{\hat G}(\bar\sigma) = Z_{\hat G}$. THEN $\mathrm{DEF}_{\bar\sigma} \longrightarrow \mathrm{PDEF}_{\bar\sigma}$ IS AN ISOM, AND BOTH FUNCTORS ARE REP'BLE

IDEA OF PF LAFFORGUE'S RECONSTRUCTION THM.

USES RESULTS OF RICHARDSON ON GIT OF $\hat G^n \xrightarrow{\pi_n} \hat G^n /\!/ \hat G$

IF $(g_1, \dots, g_n) \in \hat G^n$ IS SUCH THAT $Z_{\hat G}(g_1, \dots, g_n) = Z_{\hat G}$ AND $\overline{\langle g_1, \dots, g_n \rangle}$ IS REDUCTIVE, THEN π_n IS A $\hat G^{\mathrm{AD}}$-TORSOR IN NHD OF $\pi_n(g_1, \dots, g_n)$.

$\square$

WHEN THM 2 APPLIES, WRITE $R_{\bar\sigma} \in \mathcal{C}_{\mathcal{O}}$ FOR REPRESENTING RING OF $\mathrm{Def}_{\bar\sigma}$

LET $\mathcal{A}_{K,\mathcal{O}} = \mathcal{A}_K \cap C_c(G(K)\backslash G(\mathbb{A})/G(\widehat{\mathcal{O}}_K), \mathcal{O})$

LET $B_{K,\mathcal{O}}$ = $\mathcal{O}$-SUBALG OF $\mathrm{End}_{\mathcal{O}}(\mathcal{A}_{K,\mathcal{O}})$ GEN'D BY ALL $S_{n,\underline{\gamma},f}$

IF $\mathfrak{m} \subset B_{K,\mathcal{O}}$ MAX'L, THEN CAN FIND (AFTER ENLARGING $\mathcal{O}$ IF NECESSARY) A HOM

$$\bar\sigma_{\mathfrak{m}} : \Gamma_K \longrightarrow \widehat{G}(B_{K,\mathcal{O}}/\mathfrak{m} = k)$$

ST $f(\bar\sigma_{\mathfrak{m}}(\gamma_1), \dots, \bar\sigma_{\mathfrak{m}}(\gamma_n)) = S_{n,\underline{\gamma},f}$ mod $\mathfrak{m}$

$B_{K,\mathcal{O},\mathfrak{m}}$ CARRIES A CANONICAL PSEUDO CHAR. IF THM 2 APPLIES TO $\bar\sigma_{\mathfrak{m}}$, GET A HOM

$$R_{\bar\sigma} \longrightarrow B_{K,\mathcal{O},\mathfrak{m}}$$

$\boxed{\text{}}$

<u>THM 3</u> SUPPOSE HYP'S OF THM 2 HOLD FOR $\bar\sigma_{\mathfrak{m}}$. SUPPOSE MOREOVER THAT $\bar\sigma_{\mathfrak{m}}(\Gamma_{K(\zeta_\ell)})$ IS "$\widehat{G}$-ABUNDANT". THEN $R_{\bar\sigma} \longrightarrow B_{K,\mathcal{O},\mathfrak{m}}$ IS AN ISOM AND $\mathcal{A}_{K,\mathcal{O},\mathfrak{m}}$ IS A FREE $R_{\bar\sigma}$-MOD

<u>QUESTION</u> CAN ONE SHOW $H_{K,I,W,\mathcal{O},\mathfrak{m}}$ IS FREE OVER $R_{\bar\sigma}$?

DEFORMATION THEORY IS ALSO USEFUL TO CONSTRUCT REPS
$\sigma : \Gamma_K \longrightarrow \hat{G}(\mathcal{O})$ "KHARE–WINTENBERGER METHOD"

IDEA FIX $\bar{\sigma} : \Gamma_K \longrightarrow \hat{G}(k)$ SATISFYING HYPS OF THM 2

SUPPOSE GIVEN A FAITHFUL REP $i : \hat{G} \longrightarrow SL_N$ ST
$i \circ \bar{\sigma}$ IS ABS IRR AND $\ell > 2(N+1)$

THEN 1) $\exists$ PRES $\dfrac{\mathcal{O}[[X_1, \dots, X_g]]}{(f_1, \dots, f_g)} \cong R_{\bar{\sigma}}$

2) THERE'S AN INDUCED HOM

$$R_{i \circ \bar{\sigma}} \longrightarrow R_{\bar{\sigma}}$$

WHICH IS A FINITE RING MAP

3) $R_{i \circ \bar{\sigma}}$ IS A FINITE $\mathcal{O}$-ALG (CAN USE GAITSGORY'S PF OF DE JONG'S CONJ, OR THM 3)
(FOR SL_N)

1) + 2) + 3) $\Rightarrow$ $R_{\bar{\sigma}}$ IS A FINITE $\mathcal{O}$-ALG AND A COMPLETE INTERSECTION OVER $\mathcal{O}$ $\Rightarrow$ $R_{\bar{\sigma}}$ IS $\mathcal{O}$-FLAT

$\Rightarrow$ $\exists\ \mathcal{O}'/\mathcal{O}$ AND $R_{\bar{\sigma}} \longrightarrow \mathcal{O}'$ WHICH DETERMINES A LIFT $\sigma : \Gamma_K \longrightarrow \hat{G}(\mathcal{O}')$ LIFTING $\bar{\sigma}$

IF $\bar{\sigma}(\Gamma_K) \supset \hat{G}(\mathbb{F}_\ell)$ AND $\ell > \dim(\hat{G})$, THEN σ MUST HAVE DENSE IMAGE

§ SKETCH OF PF OF MAIN THM

GIVEN $\sigma : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$ W/ ZAR-DENSE IMAGE

USE RESULTS OF CHIN, PLACE σ IN A COMPATIBLE SYSTEM $\{\sigma_\lambda : \Gamma_K \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\lambda)\}_\lambda$ λ PLACES OF $\overline{\mathbb{Q}}$ PRIME TO q

THEN σ IS AUT $\iff$ SOME / ANY σ_λ IS

NOW USE DEF'N THEORY + OTHER TRICKS FROM # FIELD SETTING TO CONSTRUCT

1) L/K GALOIS
2) $\{\rho_\lambda : \Gamma_L \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\lambda)\}_\lambda$
3) PLACES λ_0, λ_1 ST

$$\bar{\sigma}_{\lambda_0}|_{\Gamma_L} \cong \bar{\rho}_{\lambda_0}$$

AND $\bar{\rho}_{\lambda_1}$ IS AUT'C

$$\begin{tikzcd}
 & \rho_{\lambda_1} \arrow[r] & \rho_{\lambda_0} \arrow[dr] & & & & \sigma_{\lambda_0} \arrow[r] & \sigma \\
\bar{\rho}_{\lambda_1} \arrow[ur] & & & \bar{\rho}_{\lambda_0} \arrow[r, phantom, "="] & \bar{\sigma}_{\lambda_0} \arrow[urr] & & &
\end{tikzcd}$$

AUTOMORPHY $\longrightarrow$