MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas Email/Phone: rmvargas@ciencias.unam.mx / 5104243513

Speaker's Name: Alessandra Celletti

Talk Title: Analycity domains of KAM tori in some dissipative systems

Date: 08/16/18 Time: 9:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk, A. Celletti presented a joint work with R. Calleja and R. de la llave. She considers a family of conformally symplectic maps, wich are characterized by the property that the transform a symplectic form into a multiple of itself. She presented results on the study of the perturbative expansions and the domains of analyticity in the simplectic limit of the parametrization of the quasi-periodic orbits with Diophantine frequency.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - Computer Presentations: Obtain a copy of their presentation
    - Overhead: Obtain a copy or use the originals and scan them
    - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Analyticity domains of KAM tori in some dissipative systems

Alessandra Celletti

Department of Mathematics
University of Rome Tor Vergata
–o–
Joint work with R. Calleja and R. de la Llave

MSRI, August 2018

Università di Roma
Tor Vergata

Outline

Outline

Introduction

• Kolmogorov–Arnold–Moser (KAM) theory gives results on quasi–periodic motions in non–integrable dynamical systems and in particular on the persistence of invariant tori in nearly–integrable Hamiltonian systems.

• Calleja-Celletti-de la Llave (2013-): efficient KAM theory for conformally symplectic (dissipative) systems, and other results (behavior near quasi-periodic tori, partial proof of Greene's method, concrete estimates, etc.).

Introduction

• Kolmogorov–Arnold–Moser (KAM) theory gives results on quasi–periodic motions in non–integrable dynamical systems and in particular on the persistence of invariant tori in nearly–integrable Hamiltonian systems.

• Calleja-Celletti-de la Llave (2013-): efficient KAM theory for conformally symplectic (dissipative) systems, and other results (behavior near quasi-periodic tori, partial proof of Greene's method, concrete estimates, etc.).

• Adding a dissipation to a Hamiltonian system is a very singular perturbation: the Hamiltonian admits quasi-periodic solutions with many frequencies, while a system with positive dissipation leads to attractors with few quasi-periodic solutions and needs to include drift parameters.

Introduction

• Kolmogorov–Arnold–Moser (KAM) theory gives results on quasi–periodic motions in non–integrable dynamical systems and in particular on the persistence of invariant tori in nearly–integrable Hamiltonian systems.

• Calleja-Celletti-de la Llave (2013-): efficient KAM theory for conformally symplectic (dissipative) systems, and other results (behavior near quasi-periodic tori, partial proof of Greene's method, concrete estimates, etc.).

• Adding a dissipation to a Hamiltonian system is a very singular perturbation: the Hamiltonian admits quasi-periodic solutions with many frequencies, while a system with positive dissipation leads to attractors with few quasi-periodic solutions and needs to include drift parameters.

• A KAM theory with adjustment of parameters was developed in remarkable and pioneer papers: [Moser1967], see also [Broer, Simó, etc.], with a parameter count different than in [CCL].

Introduction

• The dissipative system depends on:
- conformal factor λ (measuring the dissipation);
- drift parameter μ, which is needed in dissipative systems, since to find tori, it is not sufficient to adjust the initial conditions like in the conservative case.

• AIM: consider a *dissipative* system depending on a parameter, such that when the parameter goes to zero, the system becomes symplectic.

• Analyze the domain of analyticity in the complex parameter $\varepsilon \in \mathbb{C}$:

$$\lambda = \lambda(\varepsilon) \to \lambda(0) = 1 \ .$$

• Calleja-Celletti-de la Llave, "Domains of analyticity and Lindstedt expansions of KAM tori in some dissipative perturbations of Hamiltonian systems", Nonlinearity, vol. 30, 3151-3202 (2017).

• Calleja-Celletti-de la Llave, "Existence of whiskered KAM tori for conformally symplectic systems", Preprint (2018).

Outline

1. Introduction

2. Symplectic and Conformally Symplectic Standard Maps

3. Celestial Mechanics and the zero-dissipation limit

4. Invariant tori

5. Analyticity domains for $\eta \in \mathbb{C}$

6. Conformally symplectic limit $\lambda \to 1$

Symplectic vs. Conformally Symplectic

Definition

Let $\mathcal{M} \subseteq \mathbb{R}^d \times \mathbb{T}^d$ be a symplectic manifold with symplectic form Ω. A diffeomorphism $f_\mu : \mathcal{M} \to \mathcal{M}$ is *conformally symplectic*, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that

$$f_\mu^* \Omega = \lambda \Omega \ .$$

• The system is symplectic when $\lambda = 1$.
• λ is constant for $d \geq 2$.

Definition

A vector field X_μ is *conformally symplectic* if, denoting by L_{X_μ} the Lie derivative, there exists $\lambda : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$L_{X_\mu} \Omega = \lambda \Omega \ .$$

• The time t-flow Φ_t satisfies $(\Phi_t)^* \Omega = e^{\lambda t} \Omega$.

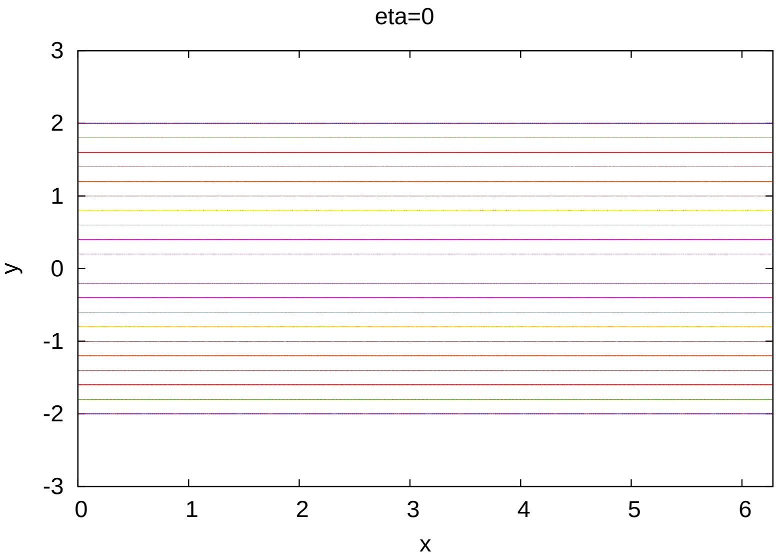
Conservative Standard Map

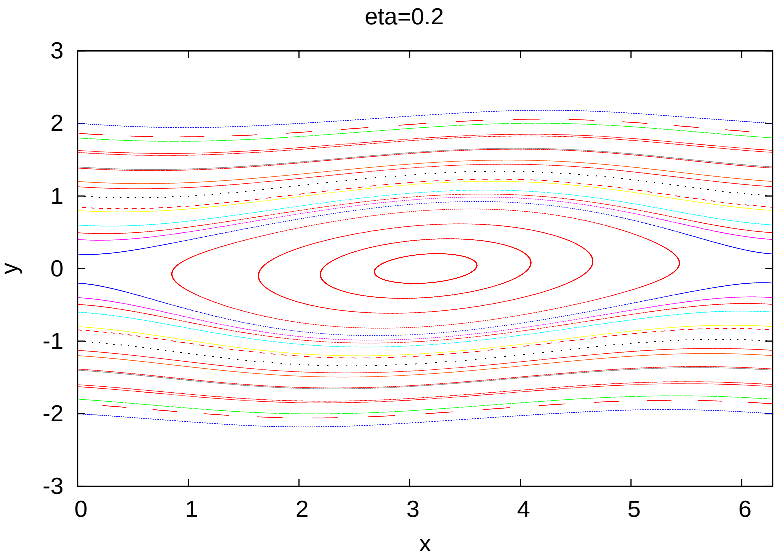
It is described by the equations (discrete analogue of the conservative spin-orbit problem):

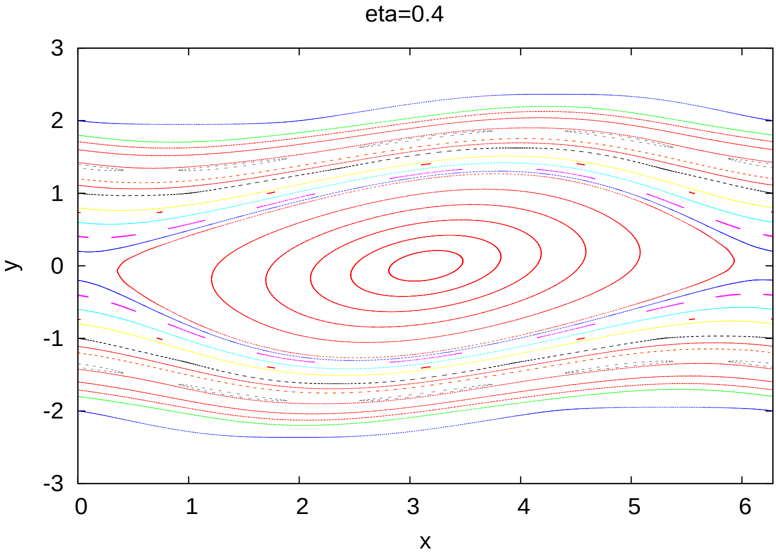
$$\begin{aligned} y' &= y + \eta \sin x \qquad\qquad y \in \mathbb{R}\,,\ x \in \mathbb{T}\,, \eta \in \mathbb{R}_+\,, \\ x' &= x + y'\,. \end{aligned}$$

• SM is integrable for $\eta = 0$, non–integrable for $\eta \neq 0$.

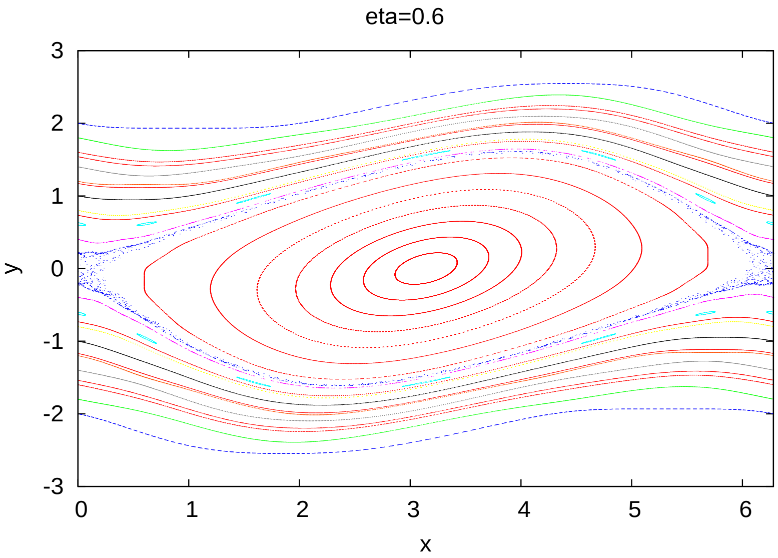
• KAM theory provides the existence of invariant curves run with quasi–periodic motions.


eta=0
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x

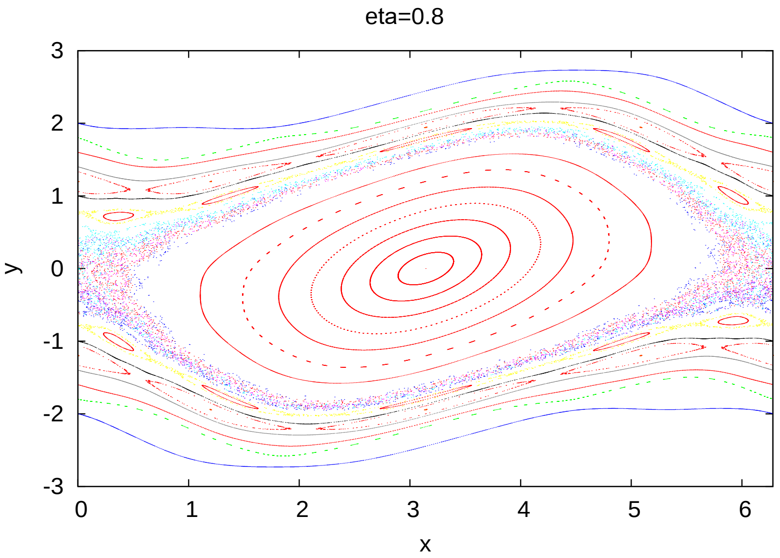

eta=0.2
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x



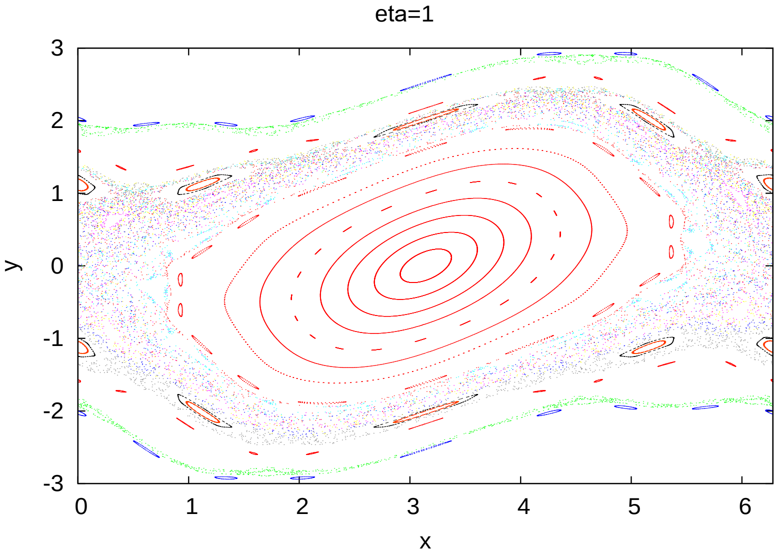
eta=0.4
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x



eta=0.6
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x



eta=0.8
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x


eta=1
3
2
1
0
-1
-2
-3
y
0
1
2
3
4
5
6
x

Dissipative Standard Map

It is described by the equations (discrete analogue of the spin-orbit problem with tidal torque):

$$\begin{aligned} y' &= \lambda y + \mu + \eta \sin x \qquad && y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, && \lambda, \eta \in \mathbb{R}_+,\ \ ,\mu \in \mathbb{R}\,, \end{aligned}$$

$0 < \lambda < 1$ dissipative parameter, μ = drift parameter.

Dissipative Standard Map

It is described by the equations (discrete analogue of the spin-orbit problem with tidal torque):

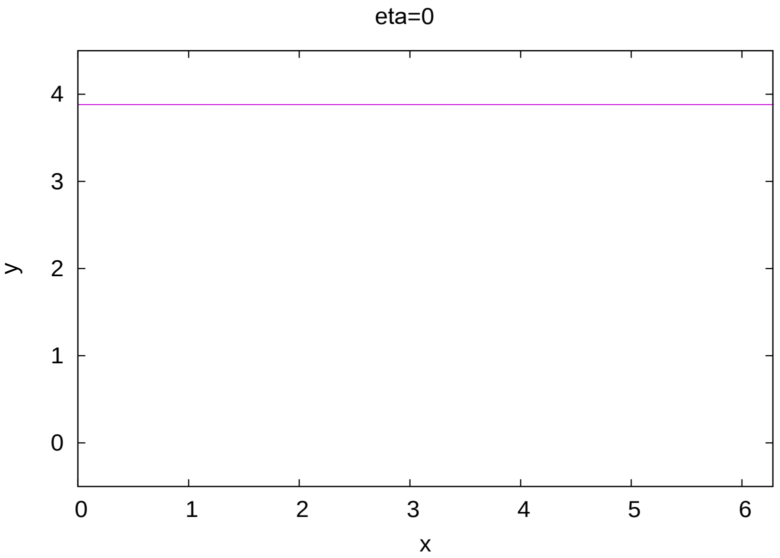
$$\begin{aligned} y' &= \lambda y + \mu + \eta \ \sin x \qquad && y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, && \lambda, \eta \in \mathbb{R}_+,\ \ ,\mu \in \mathbb{R}\,, \end{aligned}$$

$0 < \lambda < 1$ dissipative parameter, μ = drift parameter.

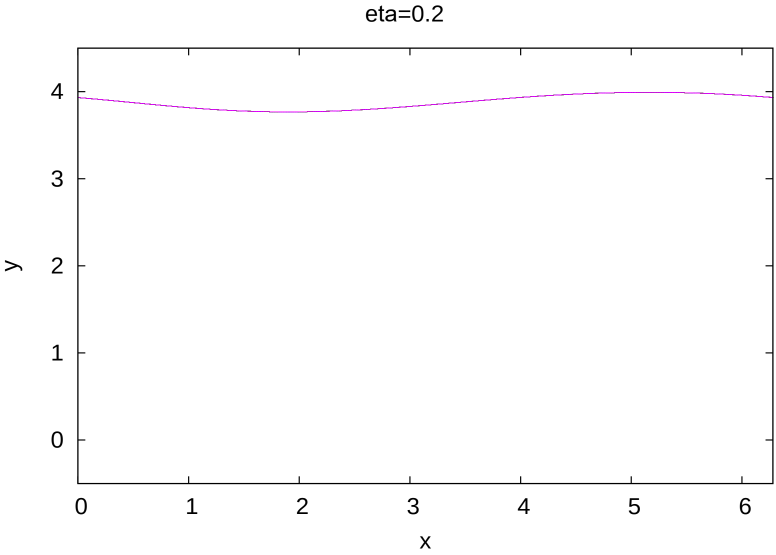
• $\lambda = 1$, $\mu = 0$ conservative SM.

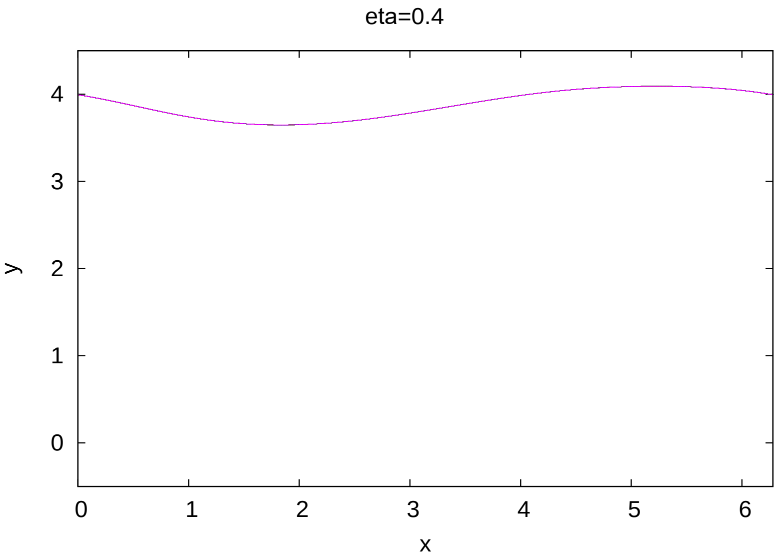
• For $\eta = 0$ the trajectory $\{y = \omega \equiv \frac{\mu}{1-\lambda}\} \times \mathbb{T}$ is invariant:

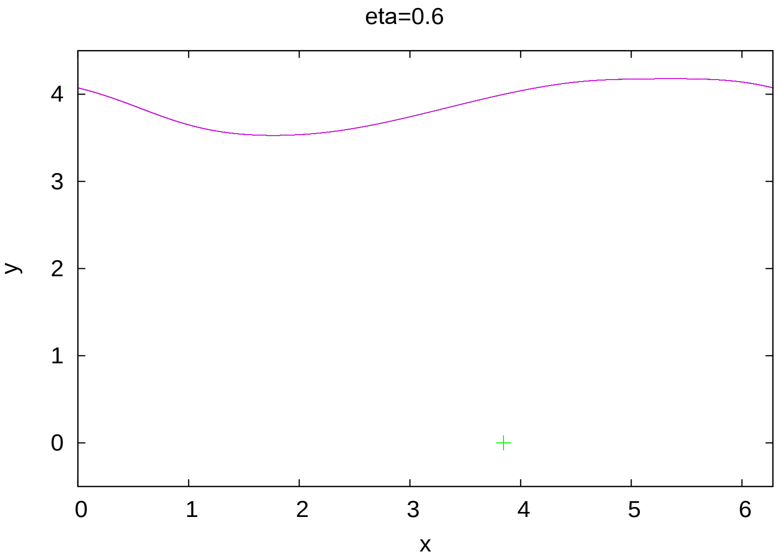
$$y' = y = \lambda y + \mu\,, \quad \omega = \lim \frac{x_j}{j} = y \implies \omega = \lambda\omega + \mu \implies \omega \equiv \frac{\mu}{1-\lambda}\,.$$

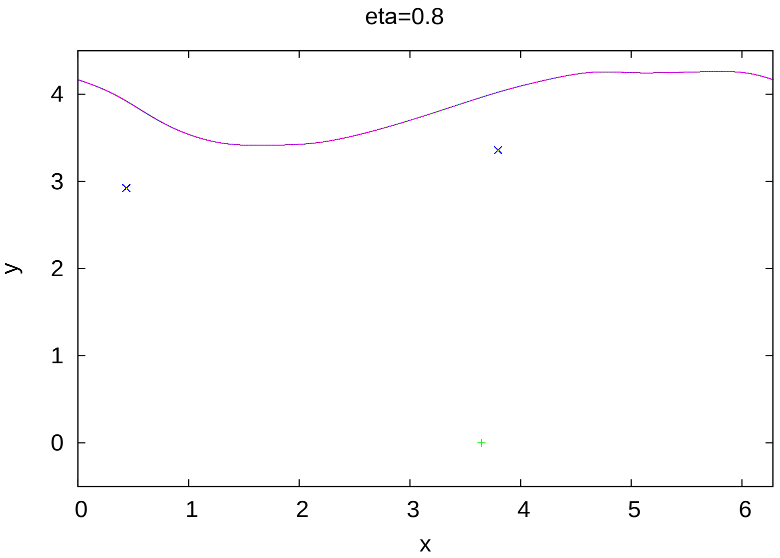


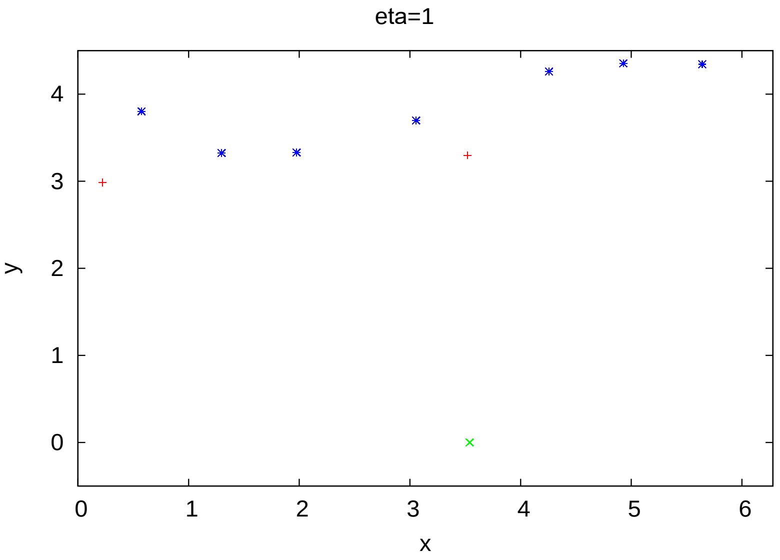
eta=0
y
4
3
2
1
0
0
1
2
3
4
5
6
x


eta=0.2
4
3
2
1
0
y
0
1
2
3
4
5
6
x


eta=0.4
y
x
0
1
2
3
4
5
6
0
1
2
3
4


eta=0.6
4
3
2
1
0
y
0
1
2
3
4
5
6
x


eta=0.8
y
x
0
1
2
3
4
5
6
0
1
2
3
4


eta=1
y
4
3
2
1
0
0
1
2
3
4
5
6
x

Outline

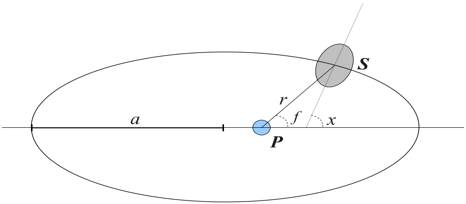
Spin–orbit model

▷ triaxial satellite $\mathcal{S}$ (with $I_1 < I_2 < I_3$);

▷ satellite moving on a Keplerian orbit around a central planet $\mathcal{P}$;

▷ spin–axis perpendicular to orbit plane and coinciding with shortest physical axis.

▷ in the dissipative case: tidal torque due to the non-rigidity of the satellite.



• Conservative equation of motion:

$$\ddot{x} + \eta \, (\frac{a}{r})^3 \sin(2x - 2f) = 0 \,, \qquad \eta = \frac{3}{2} \frac{I_2 - I_1}{I_3}$$

corresponding to a 1–dim, time–dependent Hamiltonian:

$$\mathcal{H}(y, x, t) = \frac{y^2}{2} - \frac{\eta}{2} \Big(\frac{a}{r(t)} \Big)^3 \cos(2x - 2f(t)) \,.$$

• Dissipative equation of motion: tidal torque averaged over an orbital period (e=eccentricity):

$$\ddot{x} + \eta \, \Big(\frac{a}{r} \Big)^3 \sin(2x - 2f) = -\Big(1 - \lambda(e; ...) \Big) \, \Big(\dot{x} - \mu(e) \Big) \,,$$

▷ $\lambda = \lambda(e; ...)$ plays the role of the conformal factor;
▷ $\mu = \mu(e)$ plays the role of the drift.

Zero dissipation limit

• When λ approaches 1, the dissipative system approaches the conservative one.

• We are interested to the limit $\lambda \to 1$.

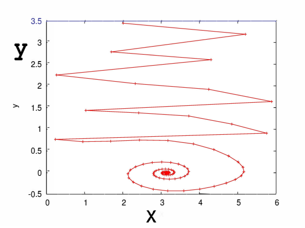
• Physical motivations:

(i) satellites rotated fast in the past, slowed down and the tidal torque becomes negligible, once the satellites have been evolved to the present state of synchronous rotation with the permanent triaxiality larger than the tidal bulge;

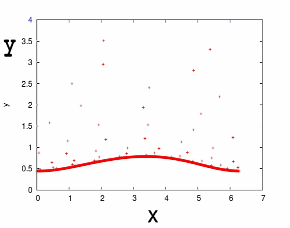
(ii) another example: the atmospheric drag is a dissipative effect on satellites in LEO, but the density of the atmosphere decreases exponentially and the dissipation becomes negligible around 2 000 km.

Role of the drift

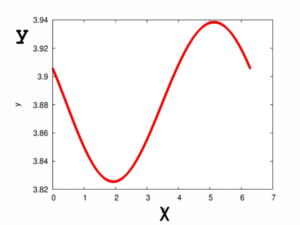
• Looking for the torus with frequency $\omega = 2\pi \, \frac{\sqrt{5}-1}{2} \simeq 3.8832$, dissipative standard map with $\eta = 0.1$, $\lambda = 0.9$.



$\mu = 0$

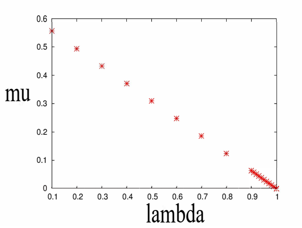


$\mu = 0.1$



$\mu = 0.0617984$

μ vs. λ



Outline

Diophantine condition

Definition

We say that the frequency vector $\omega \in \mathbb{R}^d$ satisfies the Diophantine condition if

$$|\frac{\omega}{2\pi} \cdot q - p|^{-1} \leq \nu |q|^\tau , \qquad p \in \mathbb{Z} , \quad q \in \mathbb{Z}^d \backslash \{0\} \quad \nu > 0 , \quad \tau > 0 ;$$

for $\tau > d-1$, $\mathcal{D}(\nu, \tau) =$ set of Diophantine vectors, which is of full Lebesgue measure in $\mathbb{R}^d$.

KAM surface and invariance equation

Definition

Let $\mathcal{M} \subseteq \mathbb{R}^d \times \mathbb{T}^d$ be a symplectic manifold and let $f : \mathcal{M} \to \mathcal{M}$ be a **symplectic map**. A *KAM surface* with frequency $\omega \in \mathcal{D}(\nu, \tau)$ is a d–dimensional invariant surface described parametrically by an embedding $K : \mathbb{T}^d \to \mathcal{M}$, which is the solution of the invariance equation:

$$f \circ K(\theta) = K(\theta + \omega) \ .$$

For a family f_μ of **conformally symplectic maps** depending on a real parameter μ, look for μ and an embedding K, such that

$$f_\mu \circ K(\theta) = K(\theta + \omega) \ .$$

- Symplectic case: unknown K, CS case: unknowns K, μ.

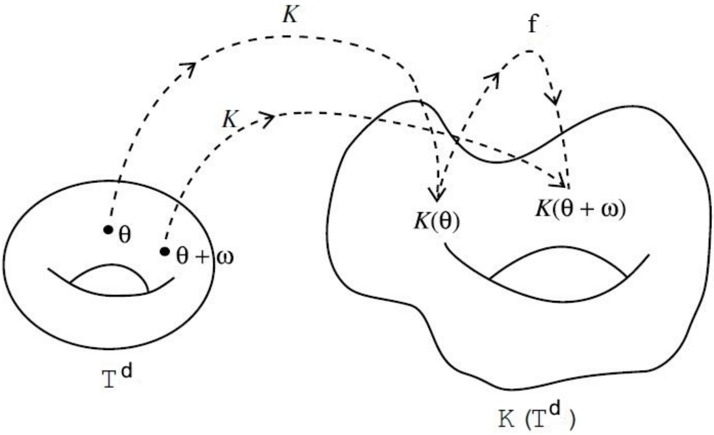


Figure: The invariance equation $f \circ K(\theta) = K(\theta + \omega)$.

Symplectic limit

- We are interested in the symplectic limit.
- Consider $\lambda = \lambda(\varepsilon)$ and a family of maps $f_{\mu,\varepsilon}$ such that

$$f^*_{\mu,\varepsilon}\,\Omega = \lambda(\varepsilon)\,\Omega\,, \qquad \lambda(0) = 1\,.$$

We discuss analyticity, so all parameters are complex.

- Symplectic limit: $\varepsilon \in \mathbb{C}$ is a small parameter that controls the dissipation:

$$\lambda(\varepsilon) = 1 + \alpha\varepsilon^a + O(|\varepsilon|^{a+1})\,, \qquad a \in \mathbb{Z}_+\,, \qquad \alpha \in \mathbb{C}\backslash\{0\}\,.$$

- Look for invariant tori by finding an embedding $K_\varepsilon : \mathbb{T}^d \to \mathcal{M}$ and a parameter vector $\mu_\varepsilon \in \mathbb{C}^d$, such that

$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon(\theta) = K_\varepsilon(\theta + \omega)\,.$$

Outline

Symplectic Standard Map $\lambda = 1$, $\mu = 0$

Analyticity domains of the conservative standard map: Berretti, Celletti, Chierchia, Falcolini, de la Llave, Marmi, Tompaidis.

• From the standard map equations:

$$x_{n+1} - x_n = y_{n+1}, \qquad x_n - x_{n-1} = y_n, \qquad y_{n+1} - y_n = \eta \sin x_n \Rightarrow$$

$$x_{n+1} - 2x_n + x_{n-1} = \eta \sin x_n \ ,$$

introduce a parametrization

$$x = \theta + u(\theta;\eta) \ ,$$

which conjugates the dynamics to a rigid rotation by ω: $\theta_{n+1} = \theta_n + \omega$.

• Compute the Lindstedt series of u:

$$u(\theta;\eta) = \sum_{j=1}^{\infty} u_j(\theta)\eta^j \ .$$

Padé approximants

• Truncated Taylor expansion at order J around 0 for a fixed θ:

$$u^{[J]}(\eta) = \sum_{j=1}^{J} u_j \eta^j \ .$$

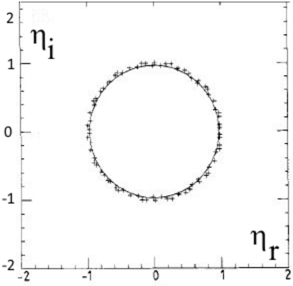
Padé approximant is a rational function, ratio of 2 polynomials, which agrees with the highest possible order with the truncated polynomial ($J = L + M$):

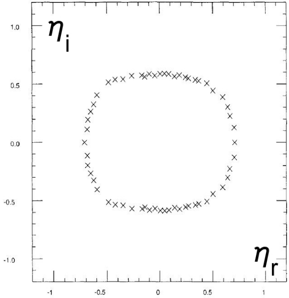
$$\frac{P_L(\eta)}{Q_M(\eta)} = \frac{a_0 + a_1\eta + ... + a_L\eta^L}{1 + b_1\eta + ... + b_M\eta^M} = U^{[L|M]}(\eta) = u^{[J]}(\eta) \ .$$

The coefficients a_j, b_j can be obtained from the condition that the first $L + M + 1$ terms coincide with the Taylor series; typically one takes diagonal Padé $L = M$:

$$u(\eta) - U^{[L|L]}(\eta) = O(\eta^{2L+1}) \ .$$

• It is believed that u has a natural boundary for $\eta \in \mathbb{C}$, i.e. its domain of analyticity is bounded by a continuous curve where singularities are dense, obstructing analytic continuation; it appears to be independent on θ (Figures by [Berretti-Chierchia, 1990] $\sin x$, golden ratio, [Berretti-Celletti-Chierchia-Falcolini, 1992] - $\sin x + \frac{1}{50} \sin 5x, \omega = [3^\infty]$).





• Greene's breakdown threshold - 0.971635 - is the intersection of the analyticity domain with the positive real axis, while the radius of convergence can be defined as

$$\rho = \inf_{\theta \in \mathbb{T}} [\limsup_{j \to \infty} |u_j(\theta)|^{\frac{1}{j}}]^{-1} \ ,$$

by studying the singularities via Padé approximants.

Conformally symplectic Standard Map $\lambda \neq 1$, $\mu \neq 0$

• Instead of using Padé approximants, compute the solution of the invariance equation, assuming $\eta \in \mathbb{C}$: applying a Newton's method, follow the solution from $\eta = 0$ increasing the real and imaginary parts of $\eta = \eta_r + i\eta_i$ until blow-up ([Calleja-Celletti 2010]).

• Using again

$$x_{n+1} - (1+\lambda)x_n + \lambda x_{n-1} - \mu = \eta \sin x_n \ ,$$

and introducing the parametrization $x = K_\mu(\theta) = \theta + u_\mu(\theta)$, expand K_μ in terms of $\eta \in \mathbb{C}$ as

$$\begin{aligned} K_\mu(\theta;\eta) &= \sum_{j=1}^{\infty} K_{\mu,j}(\theta)(\eta_r + i\eta_i)^j \\ &= K_{\mu,r}(\theta;\eta_r,\eta_i) + iK_{\mu,i}(\theta;\eta_r,\eta_i) \end{aligned}$$

$K_{\mu,j}(\theta)$ are real and the same for $g(K_\mu(\theta)) = \sin(K_\mu(\theta))$:

$$\eta\, g(K_{\mu,r} + iK_{\mu,i}) = \eta_r g_r - \eta_i g_i + i(\eta_r g_i + \eta_i g_r) \ .$$

$\Longrightarrow$ functional equation for $K_{\mu,r}$, $K_{\mu,i}$.

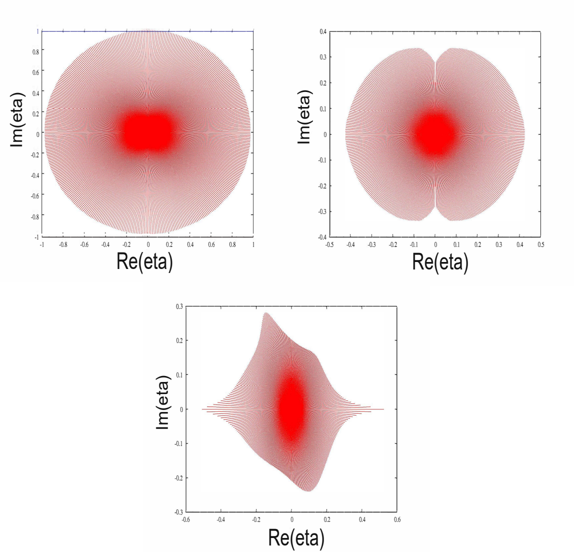


Figure: Existence domains with axes: η_r and η_i, $\lambda = 0.9$.

$a)$ $g(x) = \sin x$, $\omega/(2\pi) = \frac{\sqrt{5}-1}{2}$ (circle) - Greene's breakdown threshold = 0.97198;

$b)$ $g(x) = \sin x$, $\omega/(2\pi) = [3, 12, 1, 1, 1, 1, ...]$ (frequency close to a rational);

$c)$ $g(x) = \sin x + \frac{1}{20}\sin(4x) + \frac{1}{30}\sin(6x)$, $\omega/(2\pi) = \frac{\sqrt{5}-1}{2}$ (due to the choice of g).

Outline

Conformally symplectic limit $\lambda \to 1$

[CCL, Nonlinearity, vol. 30 (2017)]

• AIM:

▷ Study the limit $\varepsilon \to 0$, i.e. $\lambda(\varepsilon) = 1 + \alpha\varepsilon^a + O(|\varepsilon|^{a+1}) \to 1$ with $a \in \mathbb{Z}_+$, $\alpha \in \mathbb{C}\backslash\{0\}$.

▷ Study the analyticity properties of K_ε, μ_ε, namely their perturbative expansions and domains of analyticity as solutions of the invariance equation:

$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon = K_\varepsilon \circ T_\omega \ , \qquad (INV)$$

where $T_\omega(\theta) = \theta + \omega$.

Main result

• MAIN RESULT: if there exists a solution of (INV) for $\varepsilon = 0$ (symplectic case), which satisfies some mild non-degeneracy conditions, we can find K_ε, μ_ε analytic in ε for $\varepsilon \in \mathcal{G}$, where $\mathcal{G}$ is obtained *by removing from a ball centered at the origin, a sequence of (much smaller) balls with centers in smooth curves going through the origin* (see [JdlLZ99] for domains of analyticity of resonant tori in nearly-integrable systems).

• The radii of the balls decrease very fast as the centers of the excluded balls go to 0. The centers of the balls are at $|e^{ik\cdot\omega} - 1|^{1/a}$.

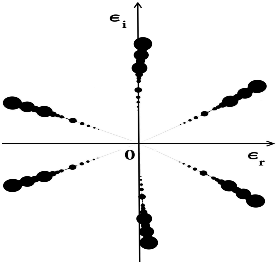


Figure: The good domain $\mathcal{G}$ is the region not covered by the black circles (the radii of the balls have been rescaled for graphical reasons); $a = 3$ ($d = 1$, $\tau = 1$).

Ingredients

$\triangleright$ An algorithm to produce a perturbative series expansion to all orders in ε, i.e. an approximate solution as a truncation solving (INV) approximately, and used as initial point of an iterative procedure which is shown to converge through...

$\triangleright$ ... a-posteriori KAM theorem for conformally symplectic systems with complex parameters: *near a nondegenerate approximately invariant torus, there is a true invariant torus* (started in [de la Llave et al 2005] for symplectic systems).

$\triangleright$ ... automatic reducibility: in the neighborhood of an invariant torus, there is a change of coordinates that makes the linearization of the invariance equation into a constant coefficient equation.

Set-up

• For $\rho > 0$, complex extension of the d–dim torus:

$$\mathbb{T}^d_\rho = \{z \in \mathbb{C}^d/\mathbb{Z}^d \ : \ \mathrm{Re}(z_j) \in \mathbb{T}\ , \quad |\mathrm{Im}(z_j)| \leq \rho\ , \quad j = 1, ..., d\}\ .$$

• $\mathcal{A}_\rho$ = vector space of functions analytic in Int($\mathbb{T}^d_\rho$), extending continuously to the boundary of $\mathbb{T}^d_\rho$.

• We endow $\mathcal{A}_\rho$ with the supremum norm, which makes it a Banach space:

$$||f||_\rho = \sup_{\theta \in \mathbb{T}^d_\rho} |f(\theta)|\ .$$

Set-up

Definition

For $\omega \in \mathbb{R}^d$, $\tau \in \mathbb{R}_+$:

$$\nu(\omega;\tau) \equiv \sup_{k\in\mathbb{Z}^d\setminus\{0\}} |e^{ik\cdot\omega} - 1|^{-1}|k|^{-\tau} .$$

If $\nu(\omega;\tau) < \infty$, we say that $\omega \in \mathbb{R}^d$ is Diophantine of class τ and constant $\nu(\omega;\tau)$.
For $\lambda \in \mathbb{C}$:

$$\nu(\lambda;\omega,\tau) \equiv \sup_{k\in\mathbb{Z}^d\setminus\{0\}} |e^{ik\cdot\omega} - \lambda|^{-1}|k|^{-\tau} .$$

If $\nu(\lambda;\omega,\tau) < \infty$, we say that λ is Diophantine with respect to ω.

- Notice that $\nu(\lambda;\omega,\tau) \le |1-|\lambda||^{-1} < \infty$ for $\lambda \neq 1$, since

$$|e^{ik\cdot\omega} - \lambda||k|^{\tau} \ge |e^{ik\cdot\omega} - \lambda| \ge \Big||e^{ik\cdot\omega}| - |\lambda|\Big| = \Big|1-|\lambda|\Big| .$$

Interlude

- We will need to solve cohomological equations of the form:

$$\begin{aligned} W_1(\theta) - W_1(\theta + \omega) &= F_1(\theta) \\ \lambda\, W_2(\theta) - W_2(\theta + \omega) &= F_2(\theta)\,. \end{aligned}$$

Expanding in Fourier series $W_j(\theta) = \sum_k \widehat{W}_{jk} e^{ik\cdot\theta}$, inserting in the equations above:

$$\begin{aligned} \sum_k \widehat{W}_{1k} e^{ik\cdot\theta}\,(1 - e^{ik\cdot\omega}) &= \sum_k \widehat{F}_{1k} e^{ik\cdot\theta} \\ \sum_k \widehat{W}_{2k} e^{ik\cdot\theta}\,(\lambda - e^{ik\cdot\omega}) &= \sum_k \widehat{F}_{2k} e^{ik\cdot\theta}\,, \end{aligned}$$

whose solution involves the small divisors

$$1 - e^{ik\cdot\omega}\,, \qquad \lambda - e^{ik\cdot\omega}\,.$$

Hence, the quantities $\nu(\omega;\tau)$, $\nu(\lambda;\omega,\tau)$.

Assumption and domains

• Assumption (**H**λ):

$$\lambda(\varepsilon) - 1 = \alpha\varepsilon^a + O(|\varepsilon|^{a+1})| , \qquad a \in \mathbb{Z}_+ , \qquad \alpha \in \mathbb{C}\backslash\{0\} .$$

• Definition of the domain $\mathcal{G}$, which is a closed set where the Diophantine constants of $\lambda(\varepsilon)$ w.r.t. ω are not too bad, so that a good approximation (up to ε^N) can be taken as initial condition for the iterative procedure:

$$\mathcal{G}(A;\omega,\tau,N) = \{\varepsilon \in \mathbb{C} : \quad \nu(\lambda(\varepsilon);\omega,\tau)\ |\lambda(\varepsilon) - 1|^{N+1} \leq A\} ,$$

and (for a typically sufficiently small) r_0:

$$\mathcal{G}_{r_0}(A;\omega,\tau,N) = \mathcal{G} \cap \{\varepsilon \in \mathbb{C} :\ |\varepsilon| \leq r_0\} .$$

• Main Theorem:

♢ Let $\mathcal{M} \equiv \mathcal{B} \times \mathbb{T}^d$, $\mathcal{B} \subseteq \mathbb{R}^d$ open, simply connected domain with smooth boundary and with symplectic form Ω; let $\omega \in \mathbb{R}^d$ be Diophantine; family of CS maps $f_{\mu,\varepsilon}$ with $\mu \in \Gamma \subseteq \mathbb{C}^d$ open; $\varepsilon \in \mathbb{C}$; conformal factor λ as in $(\mathbf{H}\lambda)$.

♢ Assume that for $\varepsilon = 0$, $f_{\mu,0}$ is symplectic and that for some μ_0 the map $f_{\mu_0,0}$ admits a Lagrangian invariant torus, i.e. we can find an analytic embedding $K_0 : \mathbb{T}^d \to \mathcal{M}$, $K_0 \in \mathcal{A}_\rho$, such that

$$f_{\mu_0,0} \circ K_0 = K_0 \circ T_\omega \ . \qquad (INV0)$$

♢ Assume that the torus K_0 satisfies a suitable non–degeneracy condition.

Statement of the main result: Theorem, Part A)

• Then, we have the following results.

A) We can find a formal power series expansion

$$K_\varepsilon^{[\infty]} = \sum_{j=0}^{\infty} \varepsilon^j K_j \ , \qquad \mu_\varepsilon^{[\infty]} = \sum_{j=0}^{\infty} \varepsilon^j \mu_j \ ,$$

satisfying (INV) in the sense of formal power series, i.e. setting

$$K_\varepsilon^{[\leq N]} = \sum_{j=0}^{N} \varepsilon^j K_j \ , \qquad \mu_\varepsilon^{[\leq N]} = \sum_{j=0}^{N} \varepsilon^j \mu_j$$

for any $N \in \mathbb{N}$ and $\rho > 0$, then for some $0 < \rho' < \rho$ and $C_N > 0$, we have

$$||f_{\mu_\varepsilon^{[\leq N]},\varepsilon} \circ K_\varepsilon^{[\leq N]} - K_\varepsilon^{[\leq N]} \circ T_\omega||_{\rho'} \leq C_N |\varepsilon|^{N+1} \ .$$

Statement of the main result: Theorem, Part B)

B) We can find a set $\mathcal{G}_{r_0} \subset \mathbb{C}$, r_0 sufficiently small, we can find

$$K_\varepsilon : \mathcal{G}_{r_0} \to \mathcal{A}_{\rho'} \ , \qquad \mu_\varepsilon : \mathcal{G}_{r_0} \to \mathbb{C}^d \ ,$$

analytic in the interior of $\mathcal{G}_{r_0}$ and extending continuously to the boundary of $\mathcal{G}_{r_0}$, such that for $\varepsilon \in \mathcal{G}_{r_0}$ they satisfy (INV) exactly:

$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon - K_\varepsilon \circ T_\omega = 0 \ .$$

Moreover, the solutions K_ε, μ_ε have the formal series of part A as asymptotic expansions for some $0 < \rho' < \rho$:

$$||K_\varepsilon^{[\leq N]} - K_\varepsilon||_{\rho'} \leq C_N |\varepsilon|^{N+1} \ , \qquad |\mu_\varepsilon^{[\leq N]} - \mu_\varepsilon| \leq C_N |\varepsilon|^{N+1} \ .$$

• REMARK: $\mathcal{G}$ is a lower bound for the analyticity domain, but we conjecture that $\mathcal{G}$ is essentially optimal in the sense that for a generic system, none of the excluded balls can be filled completely $\Rightarrow$ it is possible that the set of ε for which K_ε, μ_ε are analytic is larger than $\mathcal{G}$.

• CONSEQUENCES:

$\triangleright$ Absence of monodromy for tori, s/u bundles: one can continue uniquely along loops that enclose points outside the established domain. On the contrary, [JdlLZ99] proved no monodromy for tori, non-trivial monodromy of s/u bundles.

$\triangleright$ The functions K_ε, μ_ε are monogenic at many points in $\mathcal{G}$, i.e. points for which $\lambda(\varepsilon)$ is Diophantine w.r.t. ω.

$k = k(\varepsilon)$ is *monogenic* in a complex set if there exists the limit $\lim \frac{k(\varepsilon)-k(\varepsilon_0)}{\varepsilon-\varepsilon_0}$; when the set is open, we have the definition of differentiable function.

$\triangleright$ K_ε, μ_ε are Whitney differentiable in $\mathcal{G}$: we can find series expansion of the solution around any point in $\mathcal{G}$ and this will be the Whitney derivatives.

Proof of the main Theorem

• To prove the Theorem, we need the following result, which shows that for $\lambda \in \mathbb{C}$, given an approximate solution (later the truncated power series) satisfying a non-degeneracy condition, by an a-posteriori method we can start an iterative procedure which is shown to converge.

KAM Theorem:

$\diamondsuit$ Let $\mathcal{M} \equiv \mathcal{B} \times \mathbb{T}^d$, $\omega \in \mathbb{R}^d$ Diophantine, $\nu(\omega;\tau) < \infty$, $\nu(\lambda;\omega,\tau) < \infty$, $f_{\mu,\varepsilon}$ with $\mu \in \Gamma \subseteq \mathbb{C}^d$ (complex) conformally symplectic maps, $\varepsilon \in \mathbb{C}$, $\lambda = \lambda(\varepsilon)$ complex.

$\diamondsuit$ Let K_a, μ_a be an approximate solution of (INV) with error term E

$$f_{\mu_a,\varepsilon} \circ K_a - K_a \circ T_\omega = E \, .$$

$\diamondsuit$ Assume that a suitable non–degeneracy condition (involving λ) is satisfied:

$$\det \begin{pmatrix} \overline{S} & \overline{S\,B} + \overline{A_1} \\ (\lambda - 1)\mathrm{Id} & \overline{A_2} \end{pmatrix} \neq 0 \, .$$

Proof of the main Theorem

$\diamondsuit$ For $\mu \in \Gamma$, $f_{\mu,\varepsilon}$ is a C^1-family of analytic functions on an open connected domain $\mathcal{C} \subset \mathbb{C}^d \backslash \mathbb{Z}^d \times \mathbb{C}^d$. Assume that there exists $\zeta > 0$, so that

$$\mathrm{dist}(\mu_a, \partial\Gamma) \geq \zeta \ , \qquad \mathrm{dist}(K_a(\mathbb{T}^d), \partial\mathcal{C}) \geq \zeta \ .$$

$\diamondsuit$ Assume that the solution is sufficiently approximate, i.e. for some $0 < \delta < \rho$ and C constant:

$$||E||_\rho \leq C \left[\nu(\omega;\tau)\,\nu(\lambda;\omega,\tau)\right]^2 \delta^{4(\tau+d)} \ .$$

• Then, there exist K_ε, μ_ε, such that

$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon - K_\varepsilon \circ T_\omega = 0$$

and for positive constants C_K, C_μ:

$$\begin{aligned} ||K_\varepsilon - K_a||_{\rho-\delta} &\leq C_K\,\nu(\omega;\tau)^{-1}\,\nu(\lambda;\omega,\tau)^{-1}\,\delta^{-2(\tau+d)}\,||E||_\rho \ , \\ |\mu_\varepsilon - \mu_a| &\leq C_\mu\,||E||_\rho \ . \end{aligned}$$

Non–degeneracy condition (involving λ)

$$\det \begin{pmatrix} \overline{S} & \overline{S\,B} + \overline{A_1} \\ (\lambda-1)\mathrm{Id} & \overline{A_2} \end{pmatrix} \neq 0\,.$$

- bar denotes average w.r.t. θ
- S quantity depending on K_a, DK_a, λ, $Df_{\mu_a,\varepsilon}$
- A_1, A_2 quantities depending on K_a, DK_a, λ, $D_\mu f_{\mu_a,\varepsilon}$
- B solution of cohomology equation

$$\lambda B - B\circ T_\omega = -(A_2 - \overline{A_2})\,.$$

About the proof of the KAM theorem

• The solution of (INV) is obtained by an iterative method, where at each step we need to solve 2 cohomology equations, which involve small divisors of the form

$$|e^{ik\cdot\omega} - 1|^{-1}\ , \qquad |e^{ik\cdot\omega} - \lambda|^{-1}\ .$$

• Having fixed ω and τ, the quality factor $\nu(\omega;\tau)$ $\nu(\lambda;\omega,\tau)$ is a function only of λ. We need to identify complex domains in the ε-plane, where this quality factor is bounded uniformly.

• The cohomology equations are of the form:

$$\lambda\varphi(\theta) - \varphi(\theta+\omega) = \gamma(\theta)\ ,$$

$\gamma : \mathbb{T}^d \to \mathbb{C}$ with zero average, $\gamma \in \mathcal{A}_\rho$, $\lambda \in \mathbb{C}$; a (standard) lemma states that there exists a unique solution with zero average such that

$$\|\varphi\|_{\rho-\delta} \leq C(\tau,d)\ \nu(\lambda;\omega,\tau)\ \delta^{-\tau-d}\ \|\gamma\|_\rho$$

for $0 < \delta < \rho$.

Proof of Part A)

• Start from K_0, μ_0 exact solution of

$$f_{\mu_0,0} \circ K_0 = K_0 \circ T_\omega \ . \qquad (INV0)$$

Insert $K_\varepsilon^{[\leq N]}$, $\mu_\varepsilon^{[\leq N]}$ in (INV), expand in series of ε and equate the coefficients of same power of ε to obtain recursive relations defining K_j, μ_j.

Order 1:

$$(Df_{\mu_0,0} \circ K_0)K_1 - K_1 \circ T_\omega + (D_\mu f_{\mu_0,0} \circ K_0)\mu_1 = -D_\varepsilon f_{\mu_0,0} \circ K_0 \ .$$

Order $2 \leq j \leq N$:

$$(Df_{\mu_0,0} \circ K_0)K_j - K_j \circ T_\omega + (D_\mu f_{\mu_0,0} \circ K_0)\mu_j = F_j(K_0, ..., K_{j-1}, \mu_0, ..., \mu_{j-1}) \ ,$$

where F_j is an explicit polynomial.

Proof of Part A)

- Let $M_0 = [DK_0 \mid J^{-1} \circ K_0\, DK_0\, N]$, $N = (DK_0^T\, DK_0)^{-1}$, S_0 suitable function:

$$\underline{(Df_{\mu_0,0} \circ K_0(\theta))\, M_0(\theta)} = M_0(\theta + \omega) \begin{pmatrix} \mathrm{Id} & S_0(\theta) \\ 0 & \mathrm{Id} \end{pmatrix} . \qquad (AUX)$$

- Let $K_j(\theta) = M_0(\theta) W_j(\theta)$:

$$\underline{(Df_{\mu_0,0} \circ K_0) M_0}\, W_j - M_0 \circ T_\omega\, W_j \circ T_\omega + (D_\mu f_{\mu_0,0} \circ K_0)\mu_j = F_j(K_0, ..., K_{j-1}, \mu_0, ..., \mu_{j-1})$$

and using (AUX):

$$\underline{\underline{M_0 \circ T_\omega}} \begin{pmatrix} \mathrm{Id} & S_0(\theta) \\ 0 & \mathrm{Id} \end{pmatrix} W_j - \underline{\underline{M_0 \circ T_\omega}}\, W_j \circ T_\omega + (D_\mu f_{\mu_0,0} \circ K_0)\mu_j = F_j(K_0, ..., K_{j-1}, \mu_0, ..., \mu_{j-1}),$$

which gives for $W_j = (W_{j1}, W_{j2})$:

$$\begin{aligned} W_{j2} - W_{j2} \circ T_\omega + A_{20}\mu_j &= \widetilde{E}_{j2} \\ W_{j1} - W_{j1} \circ T_\omega + A_{10}\mu_j &= \widetilde{E}_{j1} - S_0 W_{j2} \end{aligned}$$

for suitable functions $A_{10}, A_{20}, \widetilde{E}_{j1}, \widetilde{E}_{j2}$, which can be solved under the non-degeneracy condition.

Proof of Part B)

• Start from the approximate solution $(K_\varepsilon^{[\le N]}, \mu_\varepsilon^{[\le N]})$, let $A > 0$, $\varepsilon_0 \in \mathcal{G}_{r_0}(A)$, where the cohomological equations can be solved.

• Choose ε small enough; taking $(K_a, \mu_a) = (K_\varepsilon^{[\le N]}, \mu_\varepsilon^{[\le N]})$ the error is small and for r_0 suff. small all assumptions of KAM Theorem are satisfied.

• Hence, there exists an exact solution of (INV), satisfying the bounds.

Accurate numerical computation of the domain

• From [Bustamante-Calleja (2018)], dissipative standard map with $\lambda(\varepsilon) = 1 + \varepsilon^3$, $\eta = \varepsilon$, $\omega = 2\pi\,\frac{\sqrt{5}-1}{2}$, striking results!

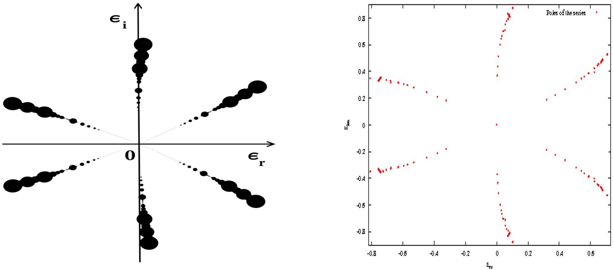


Figure: Domain $\mathcal{G}$: theoretical expectation (left) and numerical computation (right).

References.

• A. Berretti, L. Chierchia, *On the complex analytic structure of the golden-mean invariant curve for the standard map,* Nonlinearity (1990)

• A. Berretti, A. Celletti, L. Chierchia, C. Falcolini, *Natural boundaries for area-preserving twist maps,* J. Stat. Phys. (1992)

• A.P. Bustamante, R. Calleja, *Computation of domains of analyticity for the dissipative standard map in the limit of small dissipation,* Preprint (2018)

• R. Calleja, A. Celletti, *Breakdown of invariant attractors for the dissipative standard map,* CHAOS (2010)

• R. Calleja, A. Celletti, R. de la Llave, *A KAM theory for conformally symplectic systems: efficient algorithms and their validation,* J. Differential Equations **255**, n. 5 (2013)

• R. Calleja, A. Celletti, C. Falcolini, R. de la Llave, *A partial justification of Greene's criterion for conformally symplectic systems,* SIAM J. Math. Anal. **46**, n. 4 (2014)

• R. Calleja, A. Celletti, R. de la Llave, *Domains of analyticity and Lindstedt series for KAM tori in dissipative perturbations of Hamiltonian systems,* Nonlinearity (2017)

• C. Falcolini, R. de la Llave, *Numerical calculation of domains of analyticity for perturbation theories in the presence of small divisors,* J. Stat. Phys. (1992)

• A. Jorba, R. de la Llave, M. Zou, *Lindstedt series for lower-dimensional tori,* Proc. S. Agaró (1995)

• R. de la Llave, S. Tompaidis, *Computation of domains of analyticity for some perturbative expansions of mechanics,* Physica D (1994)

• R. de la Llave, S. Tompaidis, *On the singularity structure of invariant curves of symplectic mappings,* J. Nonlinear Science (1995)