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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas Email/Phone: rmvargas@ciencias.unam.mx / 5104243513

Speaker's Name: Viktor Ginzburg

Talk Title: Periodic orbits of Hamiltonian systems: the Conley conjecture and beyond

Date: 08 / 16 / 18 Time: 11 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk Viktor Ginzburg presented an overview of past and recent results around the Conley conjecture stated in 1984 and proved by Hingston in 2004. Conley conjectured that a Hamiltonian diffeomorphism of a torus has infinitely many periodic points. He also examined the situations where the Conley conjecture does not hold such as in the sphere. He also discussed the role of periodic orbits in Hamiltonian dynamics and the methods used to prove their existence

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Given by: Viktor Ginzburg

A general talk on periodic orbits of Hamiltonian systems.

- Definition
- Symplectic topological methods
- Importance
- Different perspective for the study of interesting phenomena and different aspects of dynamics.

Hamiltonian systems

- Examples
  - motion in a conservative force: $\ddot{q} = -\frac{1}{m}\nabla V(q)$
  - geodesic flows, closed geodesics. Abundance of periodic orbits.
- Class. of Hamiltonian equation

$$HE : \begin{cases} \dot{p} = \frac{-\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p}, \end{cases} \tag{0.1}$$

  Newton equation: $\ddot{q} = -\frac{1}{m}\nabla V(q)$

$$H = \frac{1}{2m} \parallel p \parallel^2 + V(q).$$

- Symplectic perspective. In Symplective Geometry, somewhat more general:we have a symplectic form. (W^{2n}, ω) symplectic manifold $\omega \in \Omega^2(W)$, $\omega^n \neq 0$, $d\omega = 0$ Example: $\omega = dp \wedge dq = \sum dp_i \wedge dq_i$, locally is always the case.

$$H : \mathbb{R} \times W \to \mathbb{R}, \ \ t \in \mathbb{R}$$

$$i_{X_H}\omega = -dH, \ \ (HE) \rightsquigarrow \varphi_H^t \text{ Hamiltonian Flow}$$

  - Ex: $\mathbb{R}^{2n}$, cotangent bundle- geodesic flow:

$$T^*M \cong TM \to \mathbb{R}$$

$$p_i q_i \ \text{ Riemannian metric } \ \frac{1}{2} < v, v >$$

  Then $\varphi^t = \varphi_H^t$ geodesic flow

Interested in periodic orbits: autonomous and time-dependent

1) $H : W \to \mathbb{R}$ autonomous.
   $\{H = c\}$ level curve fixed. Period varies.
2) $\varphi = \varphi_H = \varphi_H^1$

When $k > 1$ there are different types of orbits. We need to distinguish between:

simple *vs* iterated

The interesting case is when the system has infinitely many simple periodic orbits

A connection between the topology and the dynamics can be stated for $\mathbb{C}P^n$, $\mathbb{S}^2$, $\Sigma_{q\geq 1}$, Π^{2n} and many other symplectic manifolds.

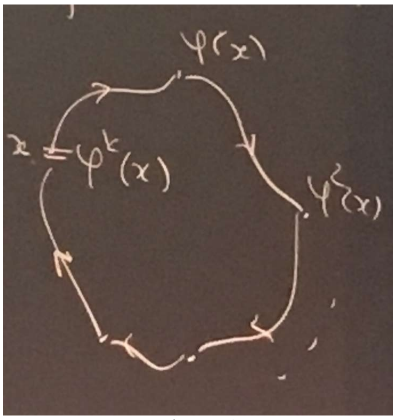

Figure 1

Variational principles Look at periodic orbits from a totally different perspective.

Meta Theorem (Least Action Principle): $k-$periodic orbits of a fixed period = critical points of the action functional A_H.
In the case of time dependant flow the definition is easy.
Simple definition: of A_H

$$A_H : \Lambda = \text{loops in} W \to \mathbb{R}$$

1) W open.

ω = dλ

ω= $d(p \wedge dq)$

$$A_H(x) = -\int_x \lambda + \int_0^k H_t(x(t)dt) \tag{0.2}$$

1) W close.

$\omega\,|_{\Pi_2(W)} = 0$

In this case I look on $A_H : \Lambda =$ Contractible loops $\to \mathbb{R}$

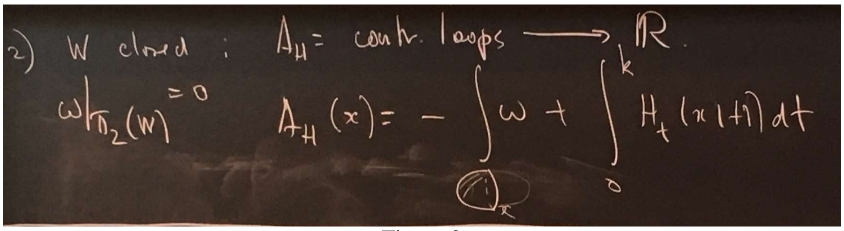

Figure 2

Rather than looking at the dynamics of φ, I look at the critical point of A_H −− > Morse Theory (non-degenerate) and **Lusternik − Schnirelmann**.

Main idea: Find $Crit(A_H) \implies$ **k-periodic orbits** This prespective translates to:

$Crit(A_H) \implies$ **complex** $C_*(H)$

$$... \to C_m(H) \to C_{m-1}(H) \to ... \tag{0.3}$$

Goal: To Use the Homology of $(C_*(H), \partial) \implies$ to study periodic orbits.

- If Homology = 0 there are no periodic orbits.
- If Homology $\neq 0 \implies$ Existence of periodic orbits. But, you don't know whether these orbits are simple or iterated!

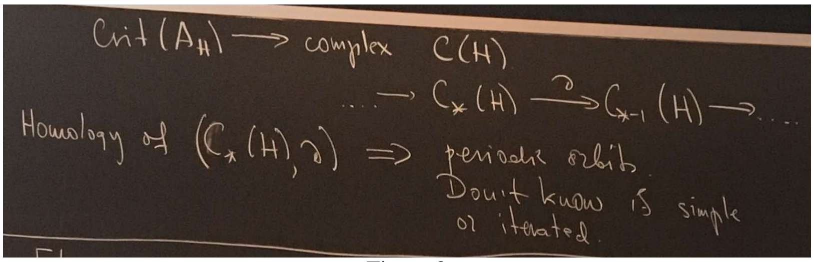

Figure 3

In symplectic topology: Floer Theory

$$\text{Floer: Floer Homology, } k=1\ HF_*(H) \cong H_*(W) \ \text{ (up to a shift of degree)}$$

$\implies$ Arnold's conjecture: lower bound on number of $1-$periodic orbits.

$$|\,\mathrm{Fix}(\varphi)\,| \geq SB(W) \tag{0.4}$$

even with non degenerate No dynamics info. It does not give any info about dynamics.; it is more a topological result. Does not capture the evolution of the system.

A step towars dynamics: Conley Conjecture Say $\varphi : (W^{2n}, \omega) \to (W^{2n}, \omega)$. And we are looking for $k-$periodic orbits

$\mathscr{P}_k(\varphi) =$ the set of all $k-$periodic orbits.

$\mathring{\mathscr{P}}_k(\varphi) =$ simple $k-$periodic orbits

I would like to understand how these sets changes depending on k. Does the set grows?

Theorem: (Conley Conjecture) For many W every φ has infinite many simple periodics orbits This conjecture has a long history and many people have worked on it: Salomon-Zehnder, Franks-Handel, Hingston,Gurel, Hein, Mazzucchelli, Ginzburg...

Also on results on the growth of the set $\mathring{\mathscr{P}}_k(\varphi)$

Floer theory Main tool but by itself is not enough since $HF(\varphi) = HF(\varphi^2) = ... = HF(\varphi^k)$ for all$k \in \mathbb{N}$

One underlying common component

Local info (e.g. a particular type of periodics orbit) $\to$ Global Homological info. $\to$ other periodic orbits and more dynamics

In Riemannian metric you have infinity many geodesics. However this infinitely many geodesics splits in two: with homological growth and no homological growth.

Counterexamples There are very simple counterexamples to the Conley-Conjecture.

Irrotational rotation of $\mathbb{S}^2$, similar to $\mathbb{CP}^n$. The dynamics is very trivial. See Figure 4

$\mathscr{P}(\varphi)$ =Poles and nothing else.

There are exactly two periodic orbits but a lot of dense orbits.

For the sake of symplicity we are still working with $\mathbb{CP}^n$

Definition of pseudo orbit (PR) $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ is a PR if $|\,\mathscr{P}\,| = n+1$, i.e, φ has as few periodic orbits as posible.

Franks theorem and pseudo-rotations Franks Theorem: Say $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ Hamiltonian diffeomorphism has $|\,Fix(\varphi)\,| \geq 3 \implies |\,\mathring{\mathscr{P}}(\varphi)\,| = \infty$

Conjecture Hofer -Zehnder: Say $\varphi : W \to W$ has more $|\,Fix(\varphi)\,|$ than necessary $\implies |\,\mathring{\mathscr{P}}(\varphi)\,| = \infty$, φ cannot be a pseudo rotation.

Conjecture Gurel, G: $\varphi : W \to W$ has a fixed point that looks out of place (unnecessary) $\implies |\,\mathring{\mathscr{P}}(\varphi)\,| = \infty$

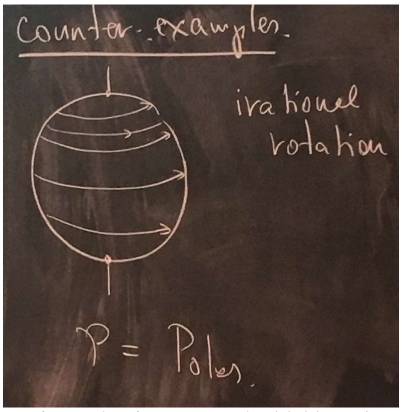

Figure 4: Figure 1 on the blakboard.

Theorem [G.,G.] : $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ has a hyperbolic fixed point $\implies |\mathring{\mathscr{P}}(\varphi)| = \infty$, φ cannot be a pseudo orbit (PO).

Theorem [G., G.] : The same is true when φ has a fixed point which is isolated and has local *homology* $\neq 0$ as an invariant set.

There are interesting examples.

Theorem $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ PR every neighborhood of any $Fix(\varphi)$ contains an entire trayectory, not isolated.

Example: For $\mathbb{CP}^1 = \mathbb{S}^2$ Le Calvez- Yoccoz-Franks.