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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas **Email/Phone:** rmvargas@ciencias.unam.mx / 5104243513

Speaker's Name: Inmaculada Baldomá

Talk Title: Invariant stable manifolds associated to parabolic objets with applications to Celestial Mechanics

Date: 08 / 16 / 18 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk I. Baldomá presented a joint work with E. Fontich and P. Martin. She presented some results on the existence and regularity of invariant manifolds (or whiskers) for either equilibrium points or invariant tori (including periodic orbits) whose normal directions are neutral in the sense that the corresponding eigenvalues are one. She showed that one can find the "whiskers" in the restricted planar n-body problem and in the full planar n-body problem.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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INVARIANT STABLE MANIFOLDS ASSOCIATED TO PARABOLIC OBJECTS WITH APPLICATIONS TO CELESTIAL MECHANICS

I. Baldomá[1], E. Fontich[2] and P. Martín[1]

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Universitat de Barcelona

Connections for Women: Hamiltonian Systems, from topology to applications through analysis, MSRI, August 16th-17th

OUTLINE

Outline

Outline

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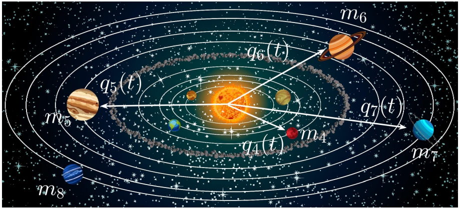
HAMILTONIAN FORMULATION

- $n+1$ point masses, evolving under the Newtonian gravitational attraction.
- Taking adequate units, the equations of motion ($F = ma$) are

$$m_i \ddot{q}_i = \sum_{j=0, i\neq j}^{n} m_i m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}.$$



- Let $p_i = m_i \dot{q}_i$, the momenta and $H(q,p) = T(p) - U(q)$ with

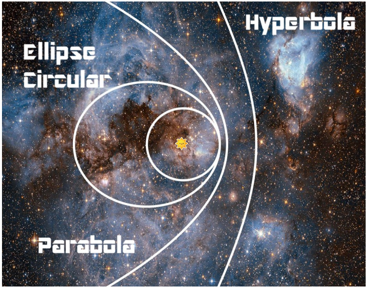
$$U(q_0, \ldots, q_n) = \sum_{0\le i<j\le n} \frac{m_i m_j}{\|q_j - q_i\|}, \qquad T(p_0, \ldots, p_n) = \sum_{i=0}^{n} \frac{1}{2m_i} p_i^2.$$

- The equations of the motion become

$$\dot{q}_i = \partial_{p_i} H, \quad \dot{p}_i = -\partial_{q_i} H, \quad \Longrightarrow \frac{dH}{dt} = \sum_i \partial_{q_i} H \dot{q}_i + \partial_{p_i} H \dot{p}_i = 0.$$

KNOWN REMARKS

2 body problem



Dimension of the problem

- If $q_j \in \mathbb{R}^3$, $6(n+1)$ unknowns, $3(n+1)$ degrees of freedom.
- If $q_j \in \mathbb{R}^2$, $4(n+1)$ unknowns, $2(n+1)$ degrees of freedom.

The restricted problem. When $m_0 = 0$.

- The motion of the primaries, bodies $1, \cdots, n$ is not affected by q_0.
- The motion of the massless body, $q = q_0$, $p = p_0$ is Hamiltonian

$$H(q,p,t) = T(p) - U(q,t) = \frac{1}{2}\|p\|^2 - \sum_{j=1}^{n} \frac{m_j}{\|q_j(t) - q\|}$$

- Either 2 and $1/2$ degrees of freedom if $q \in \mathbb{R}^2$ or 3 and $1/2$ degrees of freedom if $q \in \mathbb{R}^3$

ASYMPTOTIC STATES OF THE $n+1$ BODY PROBLEM

THE BEHAVIOR WHEN $t \to \pm\infty$ (IF THE FLOW IS COMPLETE)

Can we classify the different types of motions as $t \to \pm\infty$?

When $n+1=3$, Chazy proves that the possible asymptotic states are:

- Bounded motions.
- Hyperbolic orbits, $\|q(t)\| \to \infty$, $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
- Parabolic orbits, $\|q(t)\| \to \infty$, $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
- Oscillatory orbits, $\limsup \|q(t)\| = \infty$, $\liminf \|q(t)\| < \infty$ as $t \to \pm\infty$.

He found examples of all the motions except oscillatory orbits. The existence of such orbits have been proven by several authors: Sitnikov, Moser, Llibre-Simó, Guàrdia-Martín-Seara, etc. for some instances of the 3 body problem. Terracini also deals with parabolic orbits.

OUR GOAL

To provide necessary conditions in some $n+1$ body problems to guarantee the existence of parabolic orbits by means of the theory of invariant manifolds.

OUTLINE

EITHER MAPS OR FLOWS?

Assume that we have a vector field $\dot{x} = X(x)$. Let

$$\varphi(t; t_0, x_0) = \varphi(t - t_0; 0, x_0), \qquad \frac{d}{dt}\varphi = X(\varphi, t), \ \varphi(t_0; t_0, x_0) = x_0$$

be its flow.

Consider the map $F(x) = \varphi(1; 0, x)$:

$$F^2(x) = \varphi(1; 0, F(x)) = \varphi(2, 0, x) \Longrightarrow F^k(x) = \varphi(k, 0, x).$$

If the flow is not autonomous: $\dot{x} = X(x, t)$, we can consider instead

$$z = (x, \theta), \qquad Z(z) = (X(z), 1).$$

AS A CONCLUSION

We can extract information for flows from the information for maps.

Even when the models of celestial mechanics we deal with are vector fields, we present our result in the maps setting.

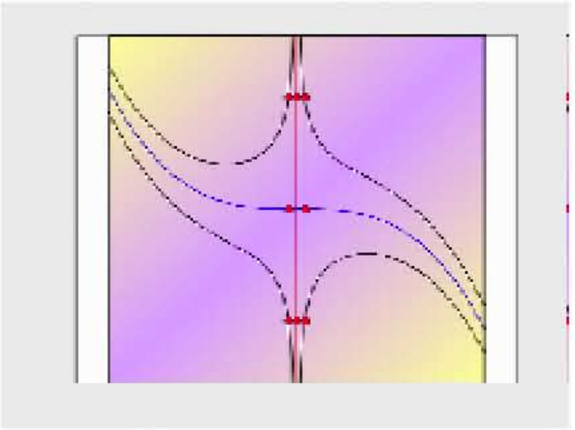
ASYMPTOTICALLY STABLE SETS FOR MAPS

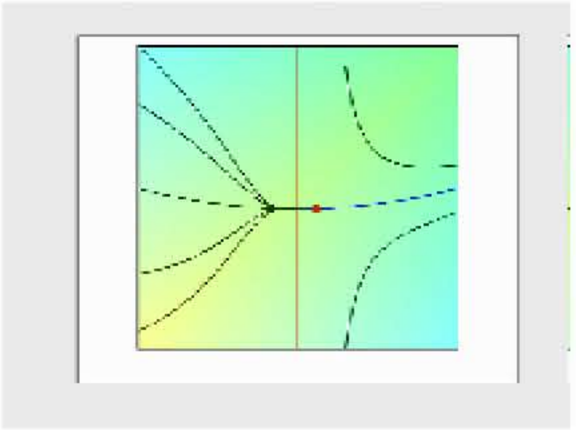
Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a real analytic map

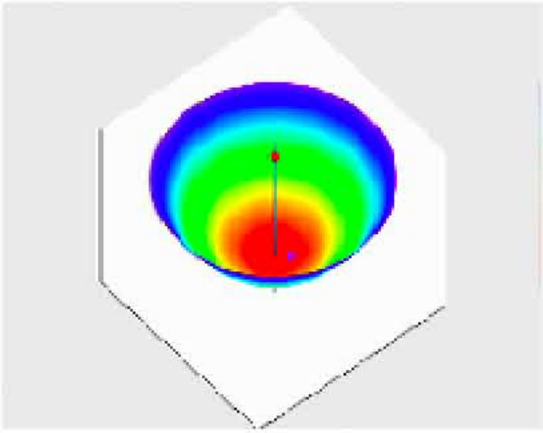
$$F(\mathcal{T}) \subset \mathcal{T}, \qquad \mathcal{T} \text{ an invariant set.}$$

Given $B \subset \mathbb{R}^n$, $\mathcal{T} \in \overline{B}$. What conditions on F we have to impose to guarantee the existence of *stable* sets

$$W_B = \{\xi \in \mathbb{R}^n : F^k(\xi) \in B, \ \forall k, \ \lim_{k\to\infty} \mathrm{dist}(\mathcal{T}, F^k(\xi)) = 0\}?.$$





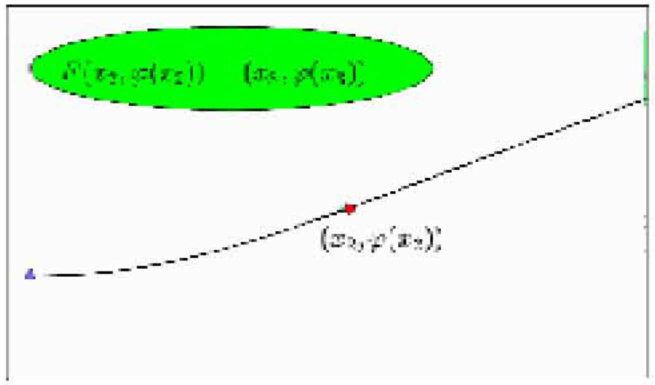


- Is W_B a manifold? of what regularity?, what is the dynamics on it?
- Can we compute an approximation?

STRATEGY TO FIND STABLE INVARIANT MANIFOLDS

Graph transform (G.T.)

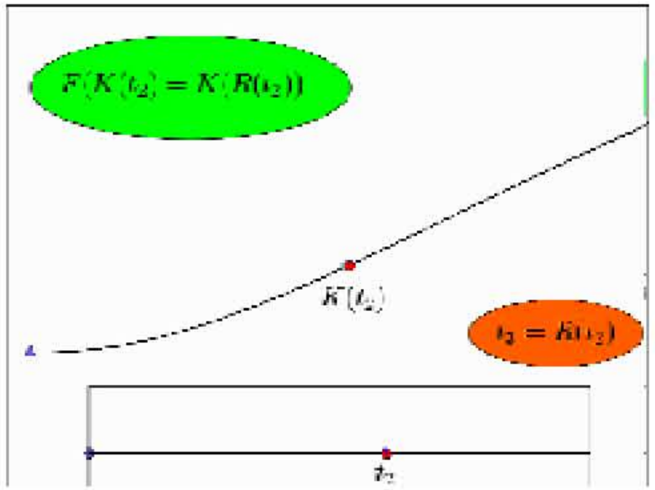
graph $\varphi \subset W_B$



See for instance [Irw70, HPS77]

Parameterization Method (P.M.)

$\{K(t)\}_t \subset W_B$



See [CFdlL03a, CFdlL03b, CFdlL05]

SOME REMARKS

- The P.M. allows more freedom than G.T which is a particular case of the P.M.
- A similar theory can be developed for vector fields $\dot{z} = X(z, t)$.
- The P.M. behaves better numerically than G.T.
- One can choose the *simplest* dynamics on W_B, denoted by R.
- The classical theory requires some nondegeneracy conditions on the dynamics around $\mathcal{T}$.

How does the parameterization method work?

- Decompose $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ being $x \in \mathbb{R}^{n_1}$ the *stable* directions.
- Perform a change of variables to ensure that the stable invariant manifold is tangent to $y = 0$.
- Find K, R solving the invariance condition $F \circ K = K \circ R$, where

$$K : V \subset \mathbb{R}^{n_1} \to \mathbb{R}^n, \qquad R : V \subset \mathbb{R}^{n_1} \to V,$$

  and $0 \in \overline{V}$. To do so,
  - *A posteriori result*. Assuming

$$F \circ K^{\leq}(x) - K^{\leq} \circ R(x) = \mathcal{O}(\|x\|^{\ell}), \ \ \ell >> 1.$$

    we prove, by using the fixed point theorem, the existence of $K^{>}$ belonging to an appropriate Banach space and satisfying

$$F \circ (K^{\leq} + K^{>}) - (K^{\leq} + K^{>}) \circ R = 0.$$

  - *An approximation result*. We provide an algorithm to compute $K^{\leq}$ and R.

OUTLINE

The parabolic infinity

Definition (if the $n+1$ bodies have trajectories defined by $t \in \mathbb{R}$)

The limit set of the one body trajectories escaping to infinity and *arriving* with zero velocity is called the parabolic infinity. We denote it by I_∞.

We have considered some instances of the $n+1$ body problem and:

- In *good* variables we prove that I_∞ is an invariant object.

If I_∞ has stable set, any trajectory in it is a parabolic orbit

- The nondegeneracy conditions needed in the classical invariant manifold theory are not satisfied.

We are forced then to

Develope a theory for invariant manifolds to degenerate invariant objects.

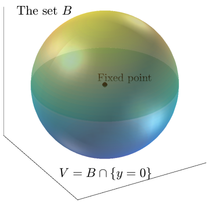
- Applying this new theory we have proved that I_∞ has a stable set (which is in fact an invariant manifold).

OUTLINE

HYPERBOLIC VERSUS PARABOLIC (I)

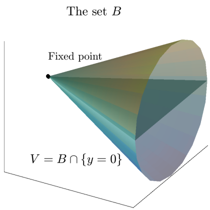
Assume that $F(0) = 0$, i.e. 0 is a fixed point.

Hyperbolic



$$F(\xi) = \mathbf{F}(\xi) + G(\xi),$$
$$\mathbf{F}(\xi) = A \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\operatorname{Spec} A \subset \{|z| \neq 1\}.$$

Parabolic (tangent to Id)



$$F(\xi) = \mathbf{F}(\xi) + G(\xi)$$
$$\mathbf{F}(\xi) = (x, y) + (p(\xi), q(\xi)),$$
$$p(x, y) = \mathcal{O}(\|(x, y)\|^N),$$
$$q(x, y) = \mathcal{O}(\|(x, y)\|^M).$$

HYPERBOLIC VERSUS PARABOLIC (II)

Hyperbolic

Let

$$A = \begin{pmatrix} A_1 & S \\ 0 & A_2 \end{pmatrix}$$

Assume that A is a hyperbolic matrix:

$\mathrm{Spec}\, A_1 \subset \{|\lambda| < 1\}$

$\mathrm{Spec}\, A_2 \subset \{|\lambda| > 1\}$

As a consequence, $\mathbf{F}(V, 0) \subset (V, 0)$.

Parabolic (tangent to Id)

Let

$$D\mathbf{F}(x, 0) = \begin{pmatrix} \mathrm{Id} + \mathbf{A}_1(x) & \mathbf{S}(x) \\ 0 & \mathrm{Id} + \mathbf{A}_2(x) \end{pmatrix}$$

Assume, for $x \in V$, the *weak hyperbolic* conditions:

$\|\mathrm{Id} + \mathbf{A}_1(x)\| \leq 1 - c_1 \|x\|^{N-1}$,

$\|\mathrm{Id} + \mathbf{A}_2(x)\| \geq 1 + c_2 \|x\|^{M-1}$.

We need to impose

$\mathrm{dist}(x + p(x, 0), \partial V) \geq c_3 \|x\|^N$.

HYPERBOLIC VERSUS PARABOLIC (III)

Hyperbolic

$K^{\leq}, R$ are polynomials:

$$K^{\leq}(x) = (x,0) + \sum_{|l|=2}^{\ell-1} K_l x^l$$

$$R(x) = A_1 x + \sum_{|l|=2}^{\ell_R} R_l x^l$$

$K = K^{\leq} + K^{>}$ is analytic in V $(0 \in V)$.

Parabolic (tangent to Id)

$K^{\leq}, R$ are sums of homogeneous functions

$$K^{\leq}(x) = (x,0) + \sum_{|l|=2}^{\ell_K} K_l(x)$$

$$R(x) = x + p(x,0) + \sum_{|l|=N+1}^{\ell_R} R_l(x)$$

$K = K^{\leq} + K^{>}$ and R can be either real analytic, $\mathcal{C}^{\infty}$ or even $\mathcal{C}^r$ on V $(0 \notin V)$.

FINAL COMMENTS

- See [CFdlL03a] and [BFM15a, BFM15b] (hyperbolic and parabolic) for more, in particular, the dependence with respect to parameters.
- Similar results for T- periodic vectors fields $X(x,y,t)$ of the form

$$X(x,y,t) = F(x,y) - (x,y) + (\mathcal{O}(\|(x,y)\|^{N+1}, \mathcal{O}(\|x,y)\|^{M+1}).$$

OUTLINE

Crazy examples. The invariant set V

Consider F, the time 1-map of

$$\dot{x}_1 = -x_1^2, \quad \dot{x}_2 = -ax_1x_2, \quad \dot{y} = bx_1y + x_2^3,$$

with $a, b > 0$. If $b + 3a < 1$, it has no stable invariant manifold. Indeed, if so then

$$y = h(x) = x_2^3 \int_{+\infty}^{0} (1 + sx_1)^{-b-3a} ds$$

which is not possible.

Weak hyperbolicity does not implies existence of manifold

In this example F satisfies the *weak hyperbolic* conditions but there is not any stable invariant set V.

Crazy examples. The loss of differentiability

Consider the map $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$,

$$F(x,y) = \begin{pmatrix} x + p(x) \\ y + 2(x_1^2 + x_2^2) + x_1^2 x_2^2 \end{pmatrix}$$

with $p(x) = (-x_1^3, -x_2^3)$. We can check that

$$\pi^y K_2(x) = x_1^2 \psi(x_2/x_1) + x_1^2 \varphi(x_2/x_1) \chi(x_2/x_1)$$

with ψ, φ analytic in $\{|x_2| < |x_1|\}$, but $\chi(z) = 2z^4 \log |z|$ is $\mathcal{C}^3$ at $z = 0$.

Analiticity of the map does not implies analyticity of the parameterization

F is analytic at 0. However, K is $\mathcal{C}^3$ in the open set $V = \{|x_2| < |x_1|\}$. With respect to $x = 0$, we can only see that

$$\lim_{x \to 0,\, x \in V} \frac{\| K(x) - (\mathrm{Id}_{n_1}, 0_{n_2})^\top \|}{\|x\|} = 0$$

The dimension of the manifold matters

In general, in the parabolic case, if $x \in \mathbb{R}^{n_1}$ with $n_1 \geq 2$:

- K and R have not Taylor expansion at the origin, even when they both are real analytic at V. In fact, only (a sort of) $\mathcal{C}^1$ regularity can be guaranteed.

However, when $x \in \mathbb{R}$, [BFdlLM07], the situation changes:

- K has Taylor expansion of any order at the origin and it is real analytic on $(0, \rho)$. Moreover, $R(x) = x - ax^N + bx^{2N-1}$, for some computable $a, b \in \mathbb{R}$. Can we say something more about the regularity at the origin in the one dimensional case?.

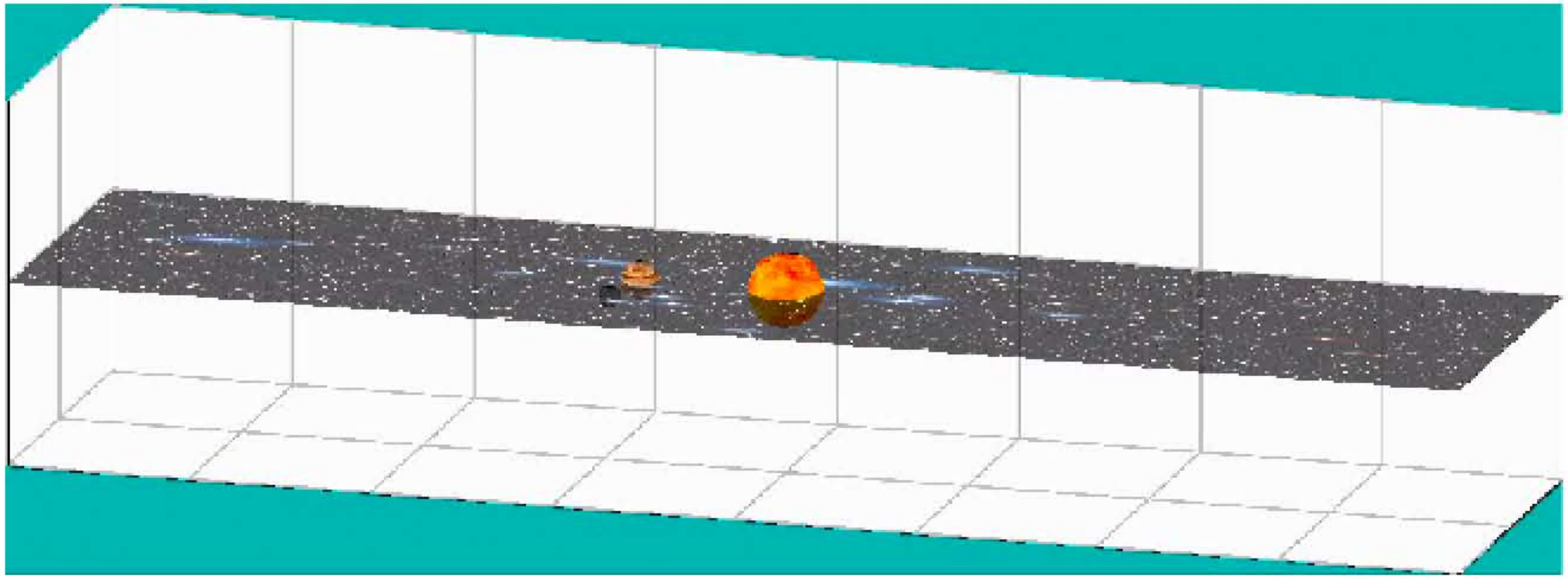
Spoiler

We will see that K is Gevrey!

OUTLINE

THE ER3BP

- Consider the restricted 3 body problem ($m_0 = 0$).
- The primaries are assumed to evolve describing Keplerian ellipses around their center of mass.
- The position q of the massless body belongs to the space, $q, p \in \mathbb{R}^3$, following a 3 and $1/2$-degrees of freedom Hamiltonian $H(q, p, t)$



PARABOLIC ORBITS?

What is the estructure of I_∞? Are there parabolic orbits?

PARABOLIC ORBITS

- Symplectic spherical coordinates (r, α, θ) and their momenta (R, A, Θ). The primaries are at $\theta = 0$.
- McGehee coordinates to bring infinity to the origin: $r = 2/u^2$.
- $I_\infty = \{u = 0,\ R = 0\}$ is folliated by periodic orbits parameterizated by $(\theta_0, \Theta_0, \alpha_0, A_0)$. We focus on the ones belonging to

$$\widehat{I}_\infty = I_\infty \cap \{\theta = \Theta = 0\}$$

- Some *singular* changes and the system satisfies our hypotheses for

$$p(x, 0) = -\frac{1}{4} x_1^4 x, \qquad \mathbf{A}_2(x) = \frac{1}{4} x_1^4 \mathrm{Id}_4, \qquad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^4.$$

EXISTENCE OF PARABOLIC ORBITS

$\widehat{I}_\infty$ has a three dimensional stable invariant manifold $W = \{K(x,t)\}$ which is $\mathcal{C}^\infty$ at the origin and real analytic in some open set V such that $0 \notin V$, $0 \in \overline{V}$. In addition $K(x,t) \sim \sum_{|l| \geq 1} K^l(t) x^l$ with $K^l(t)$ 2π-periodic.

OUTLINE

GEVREY DEFINITIONS

Formal expansions α-Gevrey

$$\hat{f}(z) = \sum_{n\geq 0} a_n z^n$$

with $|a_n| \leq c_1 c_2^n (n!)^\alpha$ or

$$|a_n| \leq c_1 c_2^n \Gamma(1+\alpha n).$$

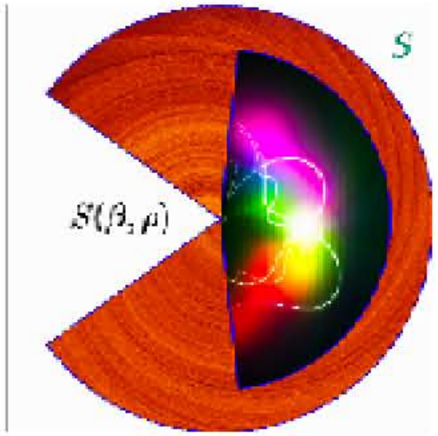
Analytic case corresponds to $\alpha = 1$.

α-Gevrey functions

$$f : S(\beta,\rho) \to \mathbb{C}$$

analytic and for any $\bar{S} \subset S(\beta,\rho)$, $\exists c_1, c_2 > 0$

$$\frac{\|D^l f(z)\|}{l!} \leq c_1 c_2^l l!^\alpha.$$



$\hat{f}$ asymptotic α-Gevrey to f, $f \equiv_\alpha \hat{f}$.

$$f : S(\beta;\rho) \to \mathbb{C}$$

analytic and for any $\bar{S} \subset S(\beta,\rho)$, $\exists c_1, c_2 > 0$

$$\left\| z^{-\ell} \left(f(z) - \sum_{l=0}^{\ell-1} a_l z^l \right) \right\| \leq c_1 c_2^\ell \ell!^\alpha$$

Notice that $f \equiv_\alpha \hat{f}$ if and only if f is α-Gevrey and

$$\lim_{z\to 0,\, z\in S} D^n f(z) = n! f_n.$$

AS A CONSEQUENCE...

$$f \equiv_\alpha \hat{0} \iff \text{for all } \bar{S} \subset S(\beta,\rho),\ \exists c,\kappa :\ |f(z)| \leq c\exp\left(-\kappa|z|^{-1/\alpha}\right),\ z \in \bar{S}$$

MAPS UNDER CONSIDERATION

Following [BH08],

- $a \neq 0$
- f_N is a homogeneous polynomial of degree $N \geq 2$ and $f_N(x,0,0) = 0$

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - a x^N + f_N(x,y,z) + f_{\geq N+1}(x,y,z) \\ y + x^{M-1}B_1 y + x^{M-1}B_2 z + g_M(x,y,z) + g_{\geq M+1}(x,y,z) \\ Cz + h_{\geq 2}(x,y,z) \end{pmatrix}$$

- g_M is a homogeneous polynomial of degree $M \geq 2$ such that $g_M(x,0,0) = 0$ and $D_{y,z} g_M(x,0,0) = 0$
- $1 \notin \operatorname{Spec} C$

MAPS UNDER CONSIDERATION

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Maps under consideration

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MAPS UNDER CONSIDERATION

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- g_M is a homogeneous polynomial of degree $M \geq 2$ such that $g_M(x,0,0) = 0$ and $D_{y,z} g_M(x,0,0) = 0$
- $1 \notin \operatorname{Spec} C$

FORMAL RESULT

We define

$$\alpha = \frac{1}{N-1}, \qquad \gamma = \begin{cases} \frac{1}{N-1}, & \text{if } N \leq M, \\ \frac{1}{N-M}, & \text{if } N > M. \end{cases}$$

THEOREM

Assume that the matrix B_1 satisfies that

- *if $M < N$, the matrix B_1 is invertible,*
- *if $M = N$, the matrices $B_1 + ka\,\mathrm{Id}$ are invertible for $k \geq 2$,*

There exist a γ-Gevrey formal expansion $\hat{K}(x) = \sum_{k\geq 1} K_k x^k$ and a polynomial $R(x) = x - ax^N + bx^{2N-1}$ such that

$$K_1 = (1,0), \quad F \circ \hat{K} - \hat{K} \circ R = \hat{0}.$$

To prove this result we use **Faà di Bruno** formula.

EXISTENCE RESULT

THEOREM

Assume that $a > 0$, and the matrices C and B_1 satisfy that

$$Spec\, C \subset \{|\lambda| > 1\}, \qquad and \qquad Spec\, B_1 \subset \{Re\lambda > 0\}.$$

Then, for any $0 < \beta < \alpha\pi$, there exist ρ small enough and a γ-Gevrey function $K : S(\beta, \rho) \to \mathbb{C}^{1+d+d'}$ such that

- *K is a solution of the invariance equation $F \circ K = K \circ R$.*
- *$K \equiv_\gamma \hat{K}$, the formal solution.*
- *If $B = B_\rho \cap \{x > 0\}$*

$$W_B = K([0, \rho)).$$

In the hyperbolic case: Spec $DF(0) \subset \{|\xi| \neq 1\}$ the parameterization K is analytic at the origin.

Borel Ritt and Banach fixed point theorem

We adapt the strategy in [BH08]:

1. We use a Borel-Ritt type result to construct a *quasi*-solution K_e:

$$F \circ K_e - K_e \circ R = E, \quad E \equiv_\gamma \hat{0}$$

   This result assures that K_e is a γ-Gevrey function.

2. For any closed sector $\bar{S}$, we prove the existence of ΔK such that

$$F \circ \left(K_e + \Delta K\right) - \left(K_e + \Delta K\right) \circ R = 0$$

   by using the fixed point theorem in the Banach space of analytic functions with the weight norm:

$$\|E\| = \max_{z \in \bar{S}} \left\| E(z) \exp\left(c|z|^{-1/\gamma}\right) \right\|.$$

Outline

EXAMPLES

We can provide examples where the Gevrey order is exactly the predicted order by our result.

- The autonomous flow, $(x, y, z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d'}$:

$$X(x, y, z) = \big(- ax^N, ax^{M-1}y + x^{M+1}b, az + x^2c \big),$$

with $a \neq 0$, $b \neq 0$ and $c \neq 0$.

- Let X be the 2π-periodic vector field defined by

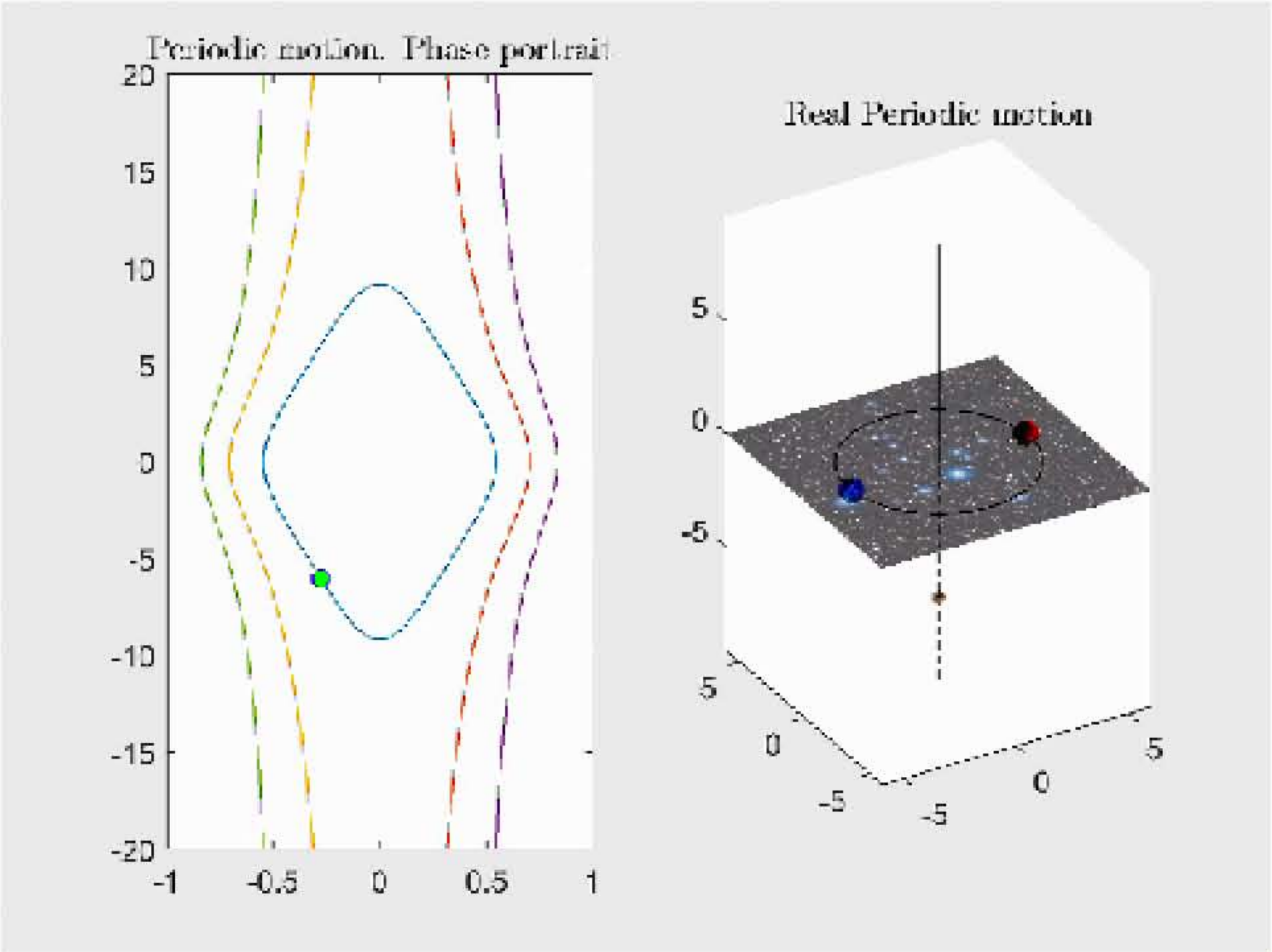
$$X(x, y, t) = \big(- ax^N, bx^{N-1}y + x^{N+1}\cos t \big), \qquad (x, y) \in \mathbb{R}^2,$$

with $a, b > 0$. The parabolic stable manifold has a formal Taylor expansion at 0 which is Gevrey of order exactly $\gamma = 1/(N-1)$.

OUTLINE

THE CONCRETE RESTRICTED THREE BODY PROBLEMS

- The two primaries move in an ellipse and the massless body moves in the line orthogonal to the plane of the primaries through their center of mass. (Sitnikov problem)
- The three body move in the same plane (R3PBP).
-



GEVREY CHARACTER OF THE PARABOLIC MANIFOLD

- Use the McGehee coordinates, [McG73] to move the infinity to the origin. This is also done in [Mos73] and [GMSS17].
- Any stroboscopic Poincaré map has the origin as a parabolic fixed point:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{1}{4}x^4 + y\mathcal{O}_3 + \mathcal{O}_5 \\ y + \frac{1}{4}x^3 y + x^2\mathcal{O}(\|y\|^2) + \mathcal{O}_5 \end{pmatrix}, \quad x \in \mathbb{R}, y \in \mathbb{R}^k.$$

USING OUR RESULTS WE PROVE THAT...

The parabolic infinity $I_\infty = \{x = 0,\ y_1 = 0\}$

- is foliated by periodic orbits.
- has an invariant manifold which is $1/3$-Gevrey.

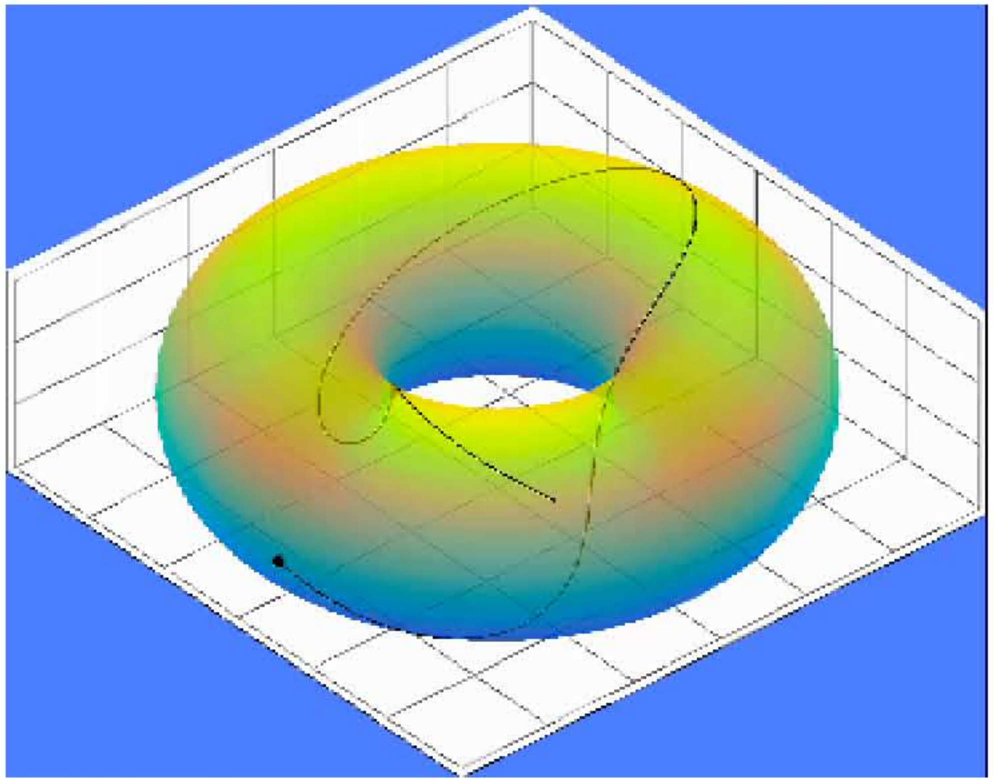
We can not prove that it is *exactly* $1/3$-Gevrey. See [MS09] where numerical computations leads to this conjecture.

Outline

INVARIANT MANIFOLDS TO INVARIANT TORI?

$F : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$.

- There is a d-torus $\mathcal{T}$ such that $F(\mathcal{T}) \subset \mathcal{T}$.
- The dynamics on the torus, $F_{|\mathcal{T}}$, is conjugated to $\theta \mapsto \theta + \omega$.



THEOREM (HYPERBOLIC CASE: *well known result*)

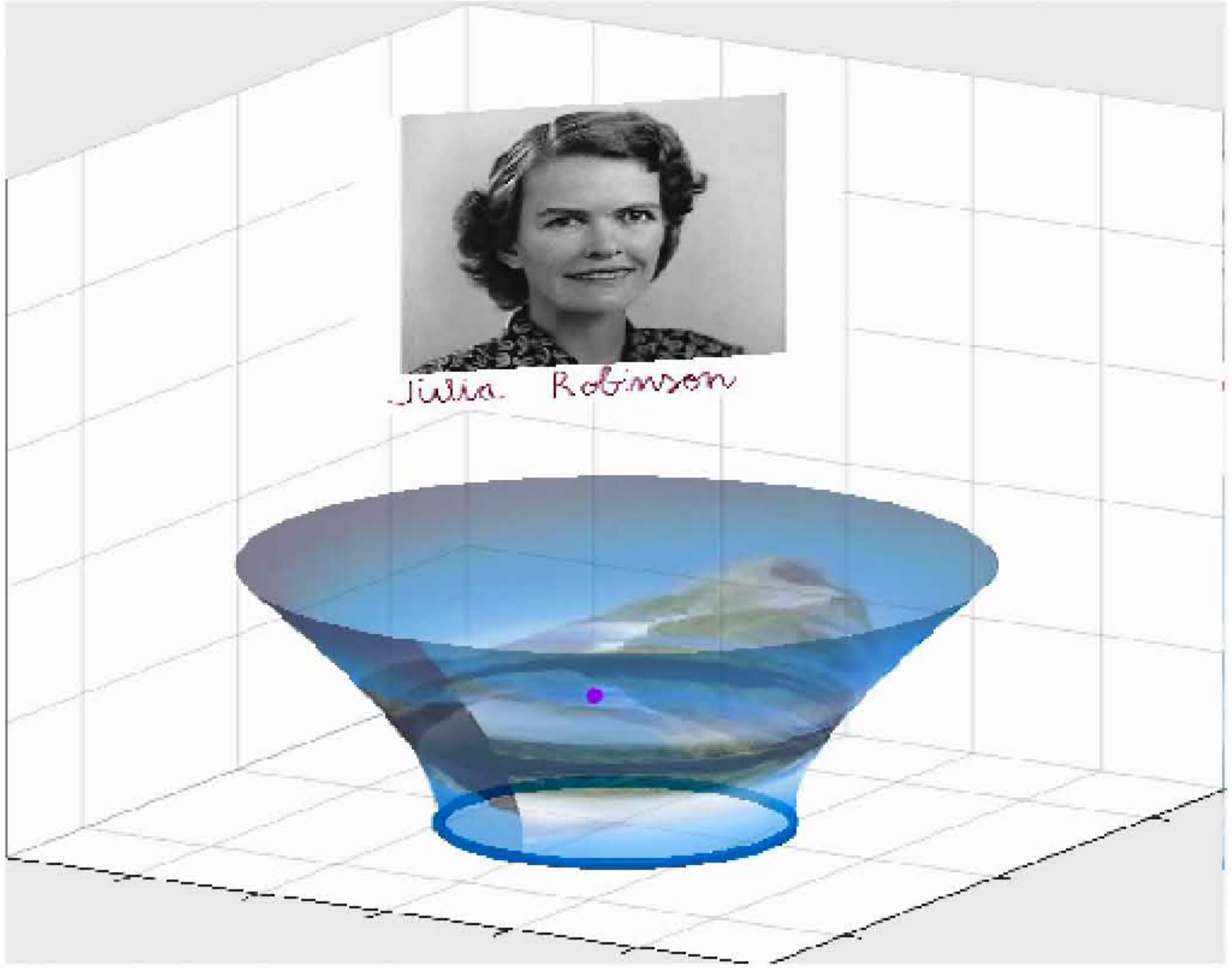
Assume F has the form

$$F(x, y, \theta, \rho) = (A^s(\theta)x, A^u(\theta)y, \theta + \Omega(x, y, \rho), \rho) + \mathcal{O}(\|(x, y, \rho)\|^2)$$

with $(x, y, \theta, \rho) \in \mathbb{R}^{n_s+n_u} \times \mathbb{T}^d \times \mathbb{R}^r$ and $\sup \|A^s(\theta)\|, \sup \|(A^u)^{-1}(\theta)\| < 1$. Then, the torus $\{x = 0,\ y = 0,\ \rho = 0\}$ has analytic stable and unstable manifolds, tangent to the x and y directions, respectively.

Connecting women

The question is, can we give sufficient conditions to connect these amazing women through stable manifolds of parabolic tori?



Partially answered...

Outline

Maps under consideration

Notation: $\overline{h} = \dfrac{1}{\mathrm{vol}(\mathbb{T}^d)} \displaystyle\int_{\mathbb{T}^d} h(\theta)\, d\theta.$

Assume that F is analytic in $\mathcal{U} \times \mathbb{T}^d$, $(0,0) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$,

- $\overline{a} > 0.$
- f_N is a homogeneous polynomial of degree $N \geq 2$ and $f_N(x,0,\theta) = 0$

$$F\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} x - a(\theta)\, x^N + f_N(x,y,\theta) + f_{\geq N+1}(x,y,\theta) \\ y + x^{N-1} B(\theta) y + g_N(x,y,\theta) + g_{\geq N+1}(x,y,\theta) \\ \theta + \omega + h_P(x,y,\theta) + h_{\geq P+1}(x,y,\theta) \end{pmatrix}$$

- $g_N(\cdot,\cdot,\theta)$ is a homogeneous polynomial of degree $N \geq 2$ such that $g_N(x,0,\theta) = 0$ and $D_y g_N(x,0,\theta) = 0$
- $h_P(\cdot,\cdot,\theta)$ is a homogeneous polynomial of degree $P \geq 1$

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The stable manifold of $\mathcal{T} = \{x = 0,\ y = 0\}$

Theorem

If ω is Diophantine, $P \geq N \geq 2$ and Spec$(\overline{B}) \subset \{\mathrm{Re}\, z > 0\}$. There exist

$$K : [0,\delta) \times \mathbb{T}^d \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{T}^d, \qquad R : [0,\delta) \times \mathbb{T}^d \to [0,\delta) \times \mathbb{T}^d :$$

1. $F \circ K = K \circ R$, $K(0,\theta) = (0,0,\theta)$, $\partial_x K(0,\theta) = (1,0,\theta)$,
$$R(x,\theta) = (x - \bar{a}x^N + bx^{2N-1}, \theta + \omega) \Longrightarrow \lim_{k\to\infty} \pi_x R^k(x,\theta) = 0.$$
2. K, R *are real analytic on* $(0,\delta) \times \mathbb{T}^d$ *and* $\mathcal{C}^\infty$ *on* $[0,\delta) \times \mathbb{T}^d$.
3. $W_{B_\rho \cap \{x>0\}} = \{K(x,\theta)\}_{(x,\theta)\in[0,\delta)\times\mathbb{T}^d}$.

Some important remarks:

- As usual, the parameterization K as well as R are not unique.
- In our proof, it is crucial for x to be unidimensional.
- Similar results can be obtained for quasiperiodic flows.

THE FIRST STEP: A POSTERIORI RESULT

A POSTERIORI RESULT

By means of the fixed point theorem we prove that, if $K^{\leq \ell}$, $R^{\leq \ell}$, $\ell \geq 2N-1$ are of the form

$$K^{\leq \ell}(x,\theta) = (x, 0, \theta + \mathcal{O}(x)) + \mathcal{O}(x^2), \qquad R^{\leq \ell}(x,\theta) = (x - \bar{a}x^N + \mathcal{O}(x^{N+1}), \theta + \omega)$$

and such that

$$F \circ K^{\leq \ell} - K^{\leq \ell} \circ R^{\leq \ell} = \mathcal{O}(x^{\ell'}), \qquad \ell' \geq \ell.$$

Then there exists a unique ΔK such that $K = K^{\leq \ell} + \Delta K$ is a solution of

$$F \circ K - K \circ R^{\leq \ell} = 0.$$

Some important remarks:

- We can keep $R = R^{\ell}$.
- We do not need here ω to be Diophantine if a and B do not depend on θ.
- We strongly use $P \geq N$ to guarantee the analyticity of the solution.
- No small divisors equations when a, B do not depend on θ.

THE SECOND STEP: APPROXIMATED SOLUTIONS

Notation: $\widetilde{h}(\theta) = h(\theta) - \overline{h}$.

THE CONSTRUCTION OF THE APPROXIMATED SOLUTION

We write $K^{\leq \ell}(x,\theta) = (x,0,\theta) + \sum_{j=2}^{\ell} K^j(\theta)x^j$, $R^{\leq \ell}(x,\theta) = x - \bar{a}x^N + \sum_{j=2}^{\ell} R^{j+N-1}(\theta)x^{j+N-1}$.

K^j, R^{j+N-1} satisfy the cohomological equations, which are, in the simplest case:

$$\widetilde{K}_y^{j+N-1}(\theta) - \widetilde{K}_y^{j+N-1}(\theta+\omega) + \widetilde{B}(\theta)\overline{K}_y^j + (\overline{B} + j\bar{a}\mathrm{Id})\overline{K}_y^j = \widetilde{E}_y^j(\theta) + \overline{E}_y^j$$

$$\widetilde{K}_x^{j+N-1}(\theta) - \widetilde{K}_x^{j+N-1}(\theta+\omega) - N\widetilde{a}(\theta)\overline{K}_x^j - \widetilde{R}_x^{j+N-1}(\theta) + (j\bar{a} - N\bar{a})\overline{K}_x^j - \overline{R}_x^{j+N-1} = \widetilde{E}_x^j(\theta) + \overline{E}_x^j$$

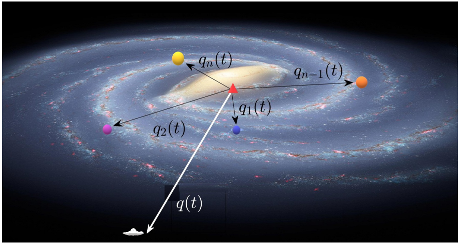
$$\widetilde{K}_\theta^{j+N-2}(\theta) - \widetilde{K}_\theta^{j+N-2}(\theta+\omega) - \widetilde{R}_\theta^{j+N-2}(\theta) + (j-1)\bar{a}\overline{K}_\theta^{j-1} - \overline{R}_\theta^{j+N-2} = \widetilde{E}_\theta^{j-1}(\theta) + \overline{E}_\theta^{j-1}$$

The functions $E_{x,y}^j, E_\theta^{j-1}$ are known. Notice that:

1. Since ω is Diophantine the equation for h, $h(\theta+\omega) - h(\theta) = \varphi(\theta)$ can be solved if $\overline{\varphi} = 0$.
2. We solve first $\overline{K}_y^j = (\overline{B} + j\bar{a}\mathrm{Id})^{-1}\overline{E}_y^j$.
3. We can take $R_\theta^{j+N-2} \equiv 0$ since $\overline{K}_\theta^{j-1} = \overline{E}_\theta^{j-1}/(j-1)$ solve the averaged equation.
4. We can take $R_x^{j+N-1} \equiv 0$, $\overline{K}_x^j = \overline{E}_x^j/(\bar{a}(j-N))$ if $j \neq N$, R_x^{2N-1} constant and $\overline{K}^{N-1}$ free.

OUTLINE

THE RESTRICTED PLANAR $n+1$-BODY PROBLEM



- The bodies move in a plane.
- Primaries move quasiperiodically

$$q_j(t) = \hat{q}_j(\omega t), \ \hat{q}_j : \mathbb{T}^d \to \mathbb{R}^2.$$

- $\|q\| >> \|q_j(t)\|$.
- 2 and $1/2$ degrees of freedom.
- Polar coordinates (r, θ) and the momenta (y, G).
- McGehee coordinates $r = \dfrac{2}{x^2}$. Transforms the 2-form into

$$-\frac{4}{x^3} dx \wedge dy + d\theta \wedge dG.$$

- Write $\phi = \omega t$ and consider its conjugated variable J. The equivalent system has $d+2$ degrees of freedom.

THE PARABOLIC INFINITY

- The system becomes

$$\dot{x}=-\frac{1}{4}x^3y,\ \dot{y}=-\frac{1}{4}x^3y,\ \dot{\theta}=\frac{1}{2}Gx^4,\ \dot{G}=\mathcal{O}(x^6),\ \dot{\phi}=\omega,\ \dot{J}=0.$$

- The d-tori $\mathcal{T}_{\theta_0,G_0}=(0,0,\theta_0,G_0,\phi)_{\phi\in\mathbb{T}^d}$ are invariant and

$$I_\infty=\bigcup_{\theta_0,G_0\in\mathbb{S}^1\times\mathbb{R}}\mathcal{T}_{\theta_0,G_0}.$$

- Fixed θ_0, G_0, making some *singular* changes, we obtain a vector field satisfying our conditions.

PARABOLIC INFINITY

If ω is Diophantine

- The d-tori $\mathcal{T}_{\theta_0,G_0}$ have $d+1$ dimensional analytic stable manifold.
- The I_∞ is a *sort of $d+2$ dimensional cylinder* having a $d+3$ dimensional analytic stable manifold.

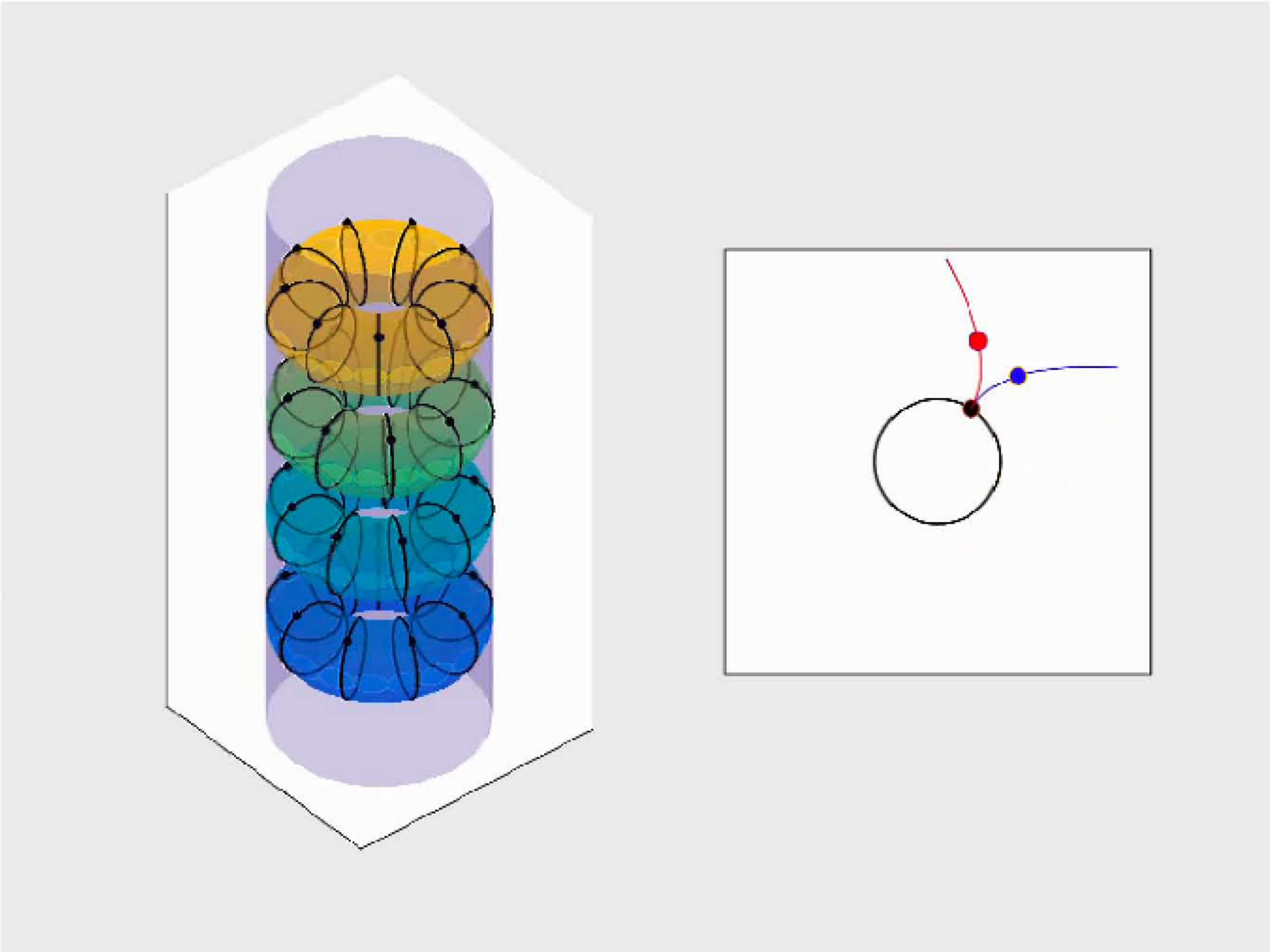
The structure of the parabolic infinity

The parabolic infinity cylinder:

$$I_\infty = \bigcup_{\theta_0, G_0 \in \mathbb{S}^1 \times \mathbb{R}} \mathcal{T}_{\theta_0, G_0}.$$

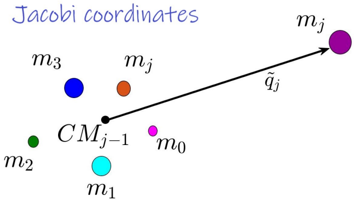
The $\mathcal{T}_{\theta_0, G_0}$ are the circles in every torus $\mathcal{T}_{G_0} = \bigcup_{\theta_0 \in \mathbb{S}^1} \mathcal{T}_{\theta_0, G_0}$.

On the right hand side, the stable (and unstable) manifold of $\mathcal{T}_{\theta_0, G_0}$.



THE PLANAR $n+1$-BODY PROBLEM

- Reduce by the linear momentum $\tilde{p}_0 = p_0 + \cdots + p_n = 0$.
- $2n$ degrees of freedom.
- Polar coordinates (r_j, θ_j) and momenta (y_j, G_j), $j = 1, \cdots, n$.
- $r_n >> r_j$ and r_j bounded.



$$CM_{j-1} = \frac{m_0 q_0 + \cdots + m_{j-1} q_{j-1}}{m_0 + \cdots + m_{j-1}}$$

- Again (and the last time) McGehee coordinates $r_n = \dfrac{2}{x_n^2}$ and:

$$\dot{r}_j = \frac{\partial \mathcal{H}}{\partial y_j},\ \dot{y}_j = -\frac{\partial \mathcal{H}}{\partial r_j} + \mathcal{O}(x_n^6),\ \dot{\theta}_j = \frac{\partial \mathcal{H}}{\partial G_j},\ \dot{G}_j = -\frac{\partial \mathcal{H}}{\partial \theta_j} + \mathcal{O}(x_n^6)$$

$$\dot{x}_n = -\frac{1}{4\mu_n} x_n^3 y_n,\ \dot{y}_n = -\frac{m_n M_n}{4} x_n^4 + \mathcal{O}(x_n^6),\ \dot{\theta}_n = \frac{1}{4\mu_n} x_n^4 G_n,\ \dot{G}_n = \mathcal{O}(x_n^6)$$

being $\mathcal{H}$ the Hamiltonian for the n-body problem.

PARABOLIC ORBITS

$$I_\infty = \bigcup_{(G_n^0,\theta_n^0)\in\mathbb{R}\times\mathbb{S}^1} \mathcal{I}_{G_n^0,\theta_n^0}, \qquad \mathcal{I}_{G_n^0,\theta_n^0} = \{x_n = y_n = 0\,,\, G_n = G_n^0,\ \theta_n = \theta_n^0\}.$$

The I_∞, is not a torus, however it has invariant tori inside. Indeed

FOR SOME VALUES OF THE MASSES, ARNOLD-FEJOZ-CHIERCHIA-PINZARI

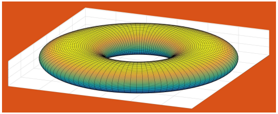
$\mathcal{I}_{G_n^0,\theta_n^0}$ has invariant Lagrangian tori $\mathcal{T}_{G_n^0,\theta_n^0}$ of dimension $2(n-1)$ with restricted dynamics $\dot\phi = \omega$, $\phi \in \mathbb{T}^{2(n-1)}$, ω Diophantine.

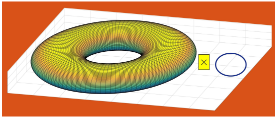
Around any $\mathcal{T}_{G_n^0,\theta_n^0}$, take action-angles variables, average and *singular* change of variables:

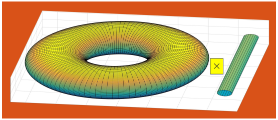
The tori $\mathcal{T}_{G_n^0,\theta_n^0}$

The tori $\mathcal{T}_{G_n^0} = \cup_{\theta_n^0\in\mathbb{S}^1}\mathcal{T}_{\theta_n^0,G_n^0}$

The cylinder $\cup_{G_n^0\in\mathbb{R}}\mathcal{T}_{G_n^0}$







$2n-1$ parabolic manifold

Lagrangian $2n$ parabolic manifold

$(2n+1)$ parabolic manifold.

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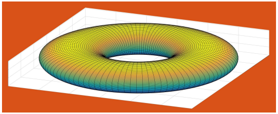

$2n-1$ parabolic manifold

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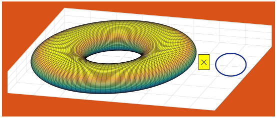
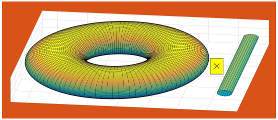


Lagrangian $2n$ parabolic manifold

The cylinder $\cup_{G_n^0\in\mathbb{R}}\mathcal{T}_{G_n^0}$



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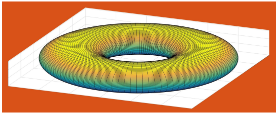

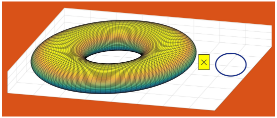

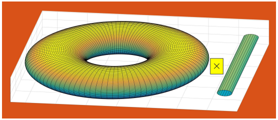

$2n-1$ parabolic manifold | Lagrangian $2n$ parabolic manifold | $(2n+1)$ parabolic manifold.

Thanks!


THANK YOU FOR YOUR ATTENTION
YESSSS, FINALLY OVER!

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YESSSS, FINALLY OVER!

THANK YOU FOR YOUR ATTENTION
YESSSS, FINALLY OVER!

THANK YOU FOR YOUR ATTENTION
YESSSS, FINALLY OVER!

THANK YOU FOR YOUR ATTENTION
YESSSS, FINALLY OVER!


This is my thank you dance!


YOUR ATTENTION
I THANK YOU FOR

YOUR ATTENTION
I THANK YOU FOR

YOUR ATTENTION
I THANK YOU FOR

YOUR ATTENTION
I THANK YOU FOR

YOUR ATTENTION
I THANK YOU FOR