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Name: Rosa Vargas **Email/Phone:** rmvargas@ciencias.unam.mx /5104243513

Speaker's Name: Susanna Terracini

Talk Title: Parabolic and collision trajectories: a survey on the variational approach to the N- body and N-centre problem.

Date: 08/16/18 **Time:** 3:30 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk S. Terracini presented the problem of the existence and the qualitative properties of parabolic and other selected trayectories (periodic, collision) for the N-body problem. She presented a review of some old and new results on the existence and classification of selected trajectories of the classical N-centre and N-body problem, with an enphasis on new analytical and geometrical techniques.

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Parabolic and collision trajectories: a survey on the variational approach to the N-body problem

Susanna Terracini

Dipartimento di Matematica "G. Peano"
Università degli Studi di Torino, Italy
ERC Advanced Grant n. 339858

Joint papers with Vivina Barutello, Alberto Boscaggin, Walter Dambrosio, Xijun Hu, Alessandro Portaluri, and Gianmaria Verzini

Connections for Women: Hamiltonian Systems,
from topology to applications through analysis
August 16, 2018 - August 17, 2018
MSRI, Berkeley

Outline

Table of Contents

References

- V. Barutello, S. Terracini and G. Verzini, *Entire Minimal Parabolic Trajectories: the planar anisotropic Kepler problem*, Arch. Rat. Mech. Anal., 207, n. 2 (2013), 583–609,
- V. Barutello, S. Terracini and G. Verzini, *Entire Parabolic Trajectories as Minimal Phase Transitions*, Calc. Var. PDE, 49 (2014), no. 1-2, 391-429,
- A. da Luz and E. Maderna, *On the free time minimizers of the newtonian n-body problem*, Math. Proc. Cambridge Philos. Soc., to appear (2011).
- N. D. Hulkower and D. G. Saari, *On the manifolds of total collapse orbits and of completely parabolic orbits for the n-body problem*, J. Differential Equations, 41 (1981), pp. 27–43.
- E. Maderna and A. Venturelli, *Globally minimizing parabolic motions in the Newtonian N-body problem*, Arch. Ration. Mech. Anal., 194 (2009), pp. 283–313.

The n-body potential

Given masses $(m_1, \dots, m_n)$, let us denote $M = \mathrm{diag}(m_1 I_d, \dots, m_n I_d)$ the mass matrix. We shall use the

$$\langle \cdot, \cdot \rangle_M = \langle M \cdot, \cdot \rangle \qquad \text{and} \qquad \|\cdot\|_M = \langle M \cdot, \cdot \rangle^{1/2} \tag{1.1}$$

respectively the Riemannian metric and the norm induced by the mass matrix. For brevity we shall refer to them respectively as the *mass scalar product* and the *mass norm*.

Let $\mathcal{X}$ denotes the *configuration space* of the n point particles with masses $m_1, \dots, m_n$ and centre of mass in 0

$$\mathcal{X} = \{(q_1, \dots, q_n) \in \mathbb{R}^{nd} | \sum_{i=1}^{n} m_i \, q_i = 0\}. \tag{1.2}$$

The collision set

Thus $\mathcal{X}$ is a N-dimensional (real) vector space, where $N = d(n-1)$. For each pair of indices $i, j \in \{1, \ldots, n\}$ let $\Delta_{i,j} = \{x \in \mathcal{X} | q_i = q_j\}$ be the *collision set of the i-th and j-th particles* and let

$$\Delta = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} \Delta_{i,j} \tag{1.3}$$

be the *collision set in* $\mathcal{X}$. It turns out that Δ is a *cone* whose vertex is the point $0 \in \Delta$; it corresponds to the *total collision* or to the *total collapse* of the system (being the centre of mass fixed at 0). The space of *collision free configurations* is denoted by

$$\widehat{\mathcal{X}} := \mathcal{X} \setminus \Delta.$$

The potential

For any real number $\alpha \in (0,2)$ and any pair $i,j \in \{1,\dots,n\}$, we consider the smooth function $V_{ij} : \mathbb{R}^d \setminus (0) \to [0,+\infty)$ given by

$$V_{ij}(z) = \frac{m_i m_j}{\|z\|^\alpha}.$$

We define the (real analytic) self-interaction *potential function* on $\widehat{\mathcal{X}}$ (the opposite of the potential energy), $V : \widehat{\mathcal{X}} \to [0,+\infty)$ as follows

$$V(q) = \sum_{\substack{i,j=1\\ i<j}}^{n} V_{ij}(q_i - q_j).$$

Associated with the potential V we have Newton's Equations of motion

$$M\,\ddot{x} = \nabla V(x)$$

where ∇ denotes the (Euclidean) gradient. As the centre of mass has an inertial motion, it is not restrictive to prescribe its position at the origin. The Lagrangian function $L : T\widehat{\mathcal{X}} \to [0, +\infty) \cup \{+\infty\}$ is given by

$$L(q, v) = \frac{1}{2}\,\|v\|_M^2 + V(q)\ ,$$

where $\|\cdot\|_M$ denotes the norm induced by the mass matrix M.

Morse Index of colliding and parabolic solutions

AIM: to develop a Morse Theory which takes into account the contribution of the interaction with the singular set Δ and the infinity

[Tanaka,K.] Non-collision solutions for a second order singular Hamiltonian systems with weak force. Ann. Inst. Henri Poincaré (1993).

[Yu,G.] Application of Morse index in weak force N-body problem. Preprint (2017).

[B.,V., Secchi,S.] Morse index properties of colliding solutions to the N-body problem. Ann. Inst. H. Poincaré (2008).

[B.,V., Hu,X., Portaluri, A., Terracini, S.] An Index Theory for asymptotic motions under singular potentials. Preprint (2017).

Zero energy trajectories

Consider the conservative dynamical system

$$\ddot{x}(t) = \nabla V(x(t)), \qquad x \in \mathbb{R}^d \setminus \Sigma, \tag{1.4}$$

where $d \geq 2$, the potential V is smooth outside –and goes to infinity near– the collision set Σ, and it satisfies the normalization condition

$$0 = \liminf_{|x| \to \infty} V(x) < V(x) \quad \text{for every } x.$$

Definition

A (global) *parabolic trajectory* for (1.4) is a collisionless solution which has null energy:

$$\frac{1}{2}|\dot{x}(t)|^2 = V(x(t)), \qquad \text{for every } t \in \mathbb{R}.$$

and moreover, $|\dot{x}(t)| \to 0$ as $t \to \pm\infty$.

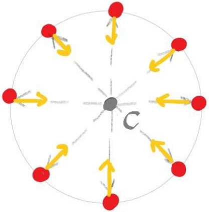
In the Kepler problem ($V(x) = 1/|x|$) all global zero-energy trajectories are indeed parabola.

Central Configurations

Definition

$s_0 \in \widehat{\mathcal{X}}$ is a central configuration if it is a critical point for the functional V constrained to the ellipsoid of inertia $\mathcal{E} = \{s \in \widehat{\mathcal{X}} : I(s) = |s|_M^2 = 1\}$.

- At a c.c. the position vector lines up with the acceleration vector (with opposite sign)
- c.c.'s generates homothethic and relative equilibria for any $\alpha \in (0, 2)$



- The classification of c.c. is still an open problem when $N \geq 4$

Asymptotic estimates in the N-body problem

As time diverges, the asymptotic behaviour of such parabolic motions has been studied by Chazy ($\sim$ 1920), Hulkover-Saari and Pollard ($\sim$ 1970). Introducing the polar coordinates

$$r := |x| > 0 \qquad s := x/|x| \in \mathbb{S}^{d-1}, \qquad \forall x \in \mathbb{R}^d \setminus \{0\},$$

if $\ddot{x} = \nabla V(x)$ on $(t_0, +\infty)$ and $r(t) \to +\infty$ as $t \to +\infty$, then the following asymptotics hold, for some $\gamma > 0$:

- $r(t) \sim (Kt)^{\frac{2}{2+\alpha}}$, as $t \to +\infty$, where $K := \frac{\alpha+2}{2}\sqrt{2\gamma}$;
- $\dot{r}(t) \sim \sqrt{2\gamma}(Kt)^{-\frac{\alpha}{2+\alpha}}$, as $t \to +\infty$;
- $\lim_{t\to+\infty} V(s(t)) = \gamma$;
- $\lim_{t\to+\infty} \nabla_T V(s(t)) = 0$;
- $\lim_{t\to+\infty} \mathrm{dist}\,(C^\gamma, s(t)) = 0$, $C^\gamma = \{s : V(s) = \gamma, \nabla_T V(s) = 0\}$,

  where $\nabla_T V(s)$ is the projection of $\nabla V(s)$ on $T_s\mathbb{S}^{d-1}$.

Motivations

In spite of their natural structural instability, these orbits act as connections between different normalized configurations and can be used as carriers from one to the other region of the phase space; as such, they have been used as building blocks for constructing complex trajectories. The existence of parabolic trajectories has been considered for the anisotropic Kepler problem and for the N-body problem;

More precisely, the presence of parabolic orbits and their variational characterization can been linked with

- the existence of minimal collision trajectories;
- the detection of unbounded families of non-collision periodic orbits;
- some applications of weak KAM theory in Celestial Mechanics; indeed, since they are homoclinic to the infinity, which represents the Aubry-Mather set of our system, they can be used to construct multiple viscosity solutions of the associated Hamilton-Jacobi equation.

A Theorem by E. Maderna and A. Venturelli

Theorem (E. Maderna, A. Venturelli 2009)

Given any initial configuration x_0 *and an asymptotic normalized central configuration* s_0 *which is a global minimum for* V *on the ellipsoid of inertia* $\mathcal{E} = \{s \in \widehat{\mathcal{X}} : I(s) = |s|_M^2 = 1\}$, *there exists a parabolic trajectory starting at* x_0 *which is asymptotic to the homotetic motion at* s_0.

Table of Contents

Morse minimizers

Given ξ^- and ξ^+ ingoing and outgoing asymptotic directions, we consider the following class of minimizers.

Definition

We say that $x \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a *(free) minimizer of $\mathcal{A}$ of parabolic type, in the sense of Morse* with asymptotic configurations $\xi^\pm$, if

- $\min_{t\in\mathbb{R}} |x(t)| > 0$;
- $|x(t)| \to +\infty$, $x(t)/|x(t)| \to \xi^\pm$ as $t \to \pm\infty$;
- for every $a < b$, $a' < b'$, and $z \in H^1(a', b')$, there holds

$$z(a') = x(a),\ z(b') = x(b) \qquad \Longrightarrow \qquad \mathcal{A}([a,b];x) \le \mathcal{A}([a',b'];z).$$

In many cases, one may be also interested in Morse minimizers in a local sense, for instance imposing some topological constraints.

Structural instability of parabolic trajectories connecting minimal central configurations

A parabolic Morse minimizer is a minimal geodesic for the Jacobi metric.

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To deal with this intrinsic structural instability we need to introduce an auxiliary parameter (the homogeneity) and look for parabolic orbits as pairs trajectory-parameter. To clarify the role of the additional parameter, it is worthwhile to let the potential vary in a class.

Definition

We denote by $\mathcal{V} = \mathcal{V}(\xi^-, \xi^+)$ the class of all α homogenous potentials sharing $\xi^{\pm}$ as minimal central configurations.

The property of a potential to admit parabolic minimizers is related to its behavior with respect to the following fixed-endpoints problem. For any $V \in \mathcal{V}$, let us define

$$c(U,\alpha) := \inf\left\{\mathcal{A}\left([a,b];x\right) : a < b,\ x(a) = \xi^-,\ x(b) = \xi^+\right\};$$

such ithe minimum is achieved by a possibly colliding solution.

Proposition

Let $U \in \mathcal{V}$; then one of the following alternatives is satisfied:

(1) $c(U,\alpha) = 4\sqrt{2U_{\min}}/(2-\alpha)$ *is achieved by the juxtaposition of two self-similar (homothetic) motions, the first connecting ξ^- to the origin and the second the origin to ξ^+;*

(2) $c(U,\alpha) < 4\sqrt{2U_{\min}}/(2-\alpha)$*, and it is achieved by trajectories which are uniformly bounded away from the origin.*

Focal behavior

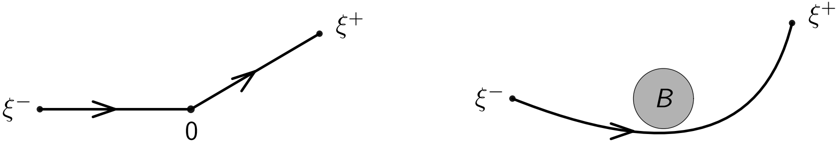
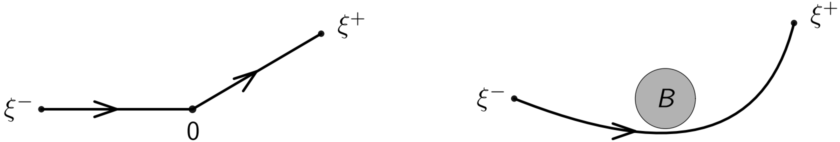


Figure: at left, $c(U,\alpha)$ is achieved by a double-homothetic motion (case (1) of the Proposition); at right $c(U,\alpha)$ is achieved by a non-collision trajectory (case (2) of the Proposition). When the second situation occurs, there exists a ball B, centered at the origin, such that any trajectory that achieves $c(U,\alpha)$ does not intersect B.

We distinguish potentials with "inner" minimizers (i.e. minimizers which pass through the origin) from potential with "outer" ones:

Focal behavior



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We distinguish potentials with "inner" minimizers (i.e. minimizers which pass through the origin) from potential with "outer" ones:

$$\mathrm{In} := \left\{(U,\alpha) \in \mathcal{V} : c(U,\alpha) = 4\sqrt{2U_{\min}}/(2-\alpha)\right\},$$

$$\mathrm{Out} := \left\{(U,\alpha) \in \mathcal{V} : c(U,\alpha) < 4\sqrt{2U_{\min}}/(2-\alpha)\right\}.$$

It is easy to see that these two sets are disjoint and their union is the whole $\mathcal{V}$; moreover, we can show that In is closed while Out is open. We are interested in their common boundary, that is

$$\Pi := \partial \mathrm{In} \cap \partial \mathrm{Out}.$$

The separating property of the common boundary is highlighted by the following result.

Lemma (Barutello, Verzini, T., 2014)

There exists an open nonempty set $\Sigma \subset \mathcal{U}$, and a continuous function $\bar{\alpha} : \Sigma \to (0,2)$ such that

$$\Pi = \{(V, \bar{\alpha}(U)) : \ U \in \Sigma\}.$$

Of course, we can exhibit explicit criteria in order to establish whether a potential $U \in \mathcal{U}$ belongs to the domain of the function $\bar{\alpha}$.

The Structure Theorem

Our main result states that the above graph coincides with the set of potentials admitting parabolic Morse minimizers.

Theorem (Barutello, Verzini, T. 2014)

$V \in \mathcal{V}$ admits a parabolic Morse minimizer if and only if $V \in \Pi$.

Of course, due to the invariance by homothetity of the problem, such Morse minimizing parabolic trajectories always come in one-parameter families and give rise to a 2-dimensional Lagrangian submanifold having boundary corresponding to the two homothetic solutions.

Remarks

In spite of their fragility, parabolic trajectories carry precious information:

- they mark a transition between different focal properties of the origin with respect to the minimal geodesics;
- are connected with the possible absence/occurrence of collisions for solutions to the Bolza problems (fixed ends);
- they can be used as carriers to travel at very low cost from one to another region of the phase space;

Remarks

In spite of their fragility, parabolic trajectories carry precious information:

- they mark a transition between different focal properties of the origin with respect to the minimal geodesics;
- are connected with the possible absence/occurrence of collisions for solutions to the Bolza problems (fixed ends);
- they can be used as carriers to travel at very low cost from one to another region of the phase space;
- As remarked by de Luz and Maderna, the property to be collisionless for all Bolza minimizers implies the absence of parabolic trajectories which are Morse minimal for the usual n–body problem with $\alpha = 1$ (without topological constraints). This can be easily seen by a rescaling and limiting argument.
- In contrast, minimal parabolic arcs (i.e., defined only on the half line) exist for every starting configuration, as proven recently by Maderna and Venturelli.

Table of Contents

Devaney's Work: the Anisotropic Kepler Problem in $\mathbb{R}^2$

In 1978 R.L. Devaney (*Invent. Math., 45*) considered the planar anisotropic Kepler potential

$$V(r\cos\vartheta, r\sin\vartheta) = \frac{U(\vartheta)}{r^\alpha}, \qquad \vartheta \in \mathbb{R},\ r > 0,$$

where U is a 2π-periodic function such that $U(\vartheta) \geq U_{\min} > 0$, $\forall \vartheta \in \mathbb{R}$.

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In 1978 R.L. Devaney (*Invent. Math., 45*) considered the planar anisotropic Kepler potential

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where U is a 2π-periodic function such that $U(\vartheta) \geq U_{\min} > 0$, $\forall \vartheta \in \mathbb{R}$.

Following Devaney, a remarkable variant of MacGehee coordinates makes the parabolic motion equations equivalent to a planar first order system. Let

$$z = \sqrt{2U(\vartheta)}$$

and, assuming $x = re^{i\theta}$, $\dot{x} = r^{-\alpha/2}ze^{i\varphi}$, introduce the new parameter τ as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = zr^{1+\alpha/2}$$

Now rewrite the dynamical system as (here “′” denotes the derivative with respect to τ)

$$\begin{cases} r' = rz^2\cos(\varphi - \vartheta) = 2rU(\vartheta)\cos(\varphi - \vartheta) \\ z' = zU'(\vartheta)\sin(\varphi - \vartheta) \\ \vartheta' = z^2\sin(\varphi - \vartheta) = 2U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta), \end{cases}$$

The above system contains the independent planar system

$$\begin{cases}\vartheta' = 2U(\vartheta)\sin(\varphi - \vartheta) \\ \varphi' = U'(\vartheta)\cos(\varphi - \vartheta) + \alpha U(\vartheta)\sin(\varphi - \vartheta).\end{cases}$$

- Stationary points: (ϑ^*, φ^*), where $U'(\vartheta^*) = 0$ and $\varphi^* = \vartheta^* + h\pi$, for some $h \in \mathbb{Z}$. Minima of U correspond to saddles, maxima to sinks/sources. Other trajectories: heteroclinics between the above.

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- The function

$$v(\tau) = \sqrt{U(\vartheta(\tau))}\cos\left(\varphi(\tau) - \vartheta(\tau)\right),$$

is non-decreasing on the solutions.

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- The function

$$v(\tau) = \sqrt{U(\vartheta(\tau))}\cos\left(\varphi(\tau) - \vartheta(\tau)\right),$$

  is non-decreasing on the solutions.
- The corresponding solutions of the systems are:
  global and unbounded if $\cos(\varphi - \vartheta) \to \pm 1$ as $\tau \to \pm\infty$;
  colliding in finite t if $\cos(\varphi - \vartheta) \to \mp 1$ as $\tau \to \pm\infty$
  (in the future/past).

Corollary

Let $\vartheta^- < \vartheta^+$ *belong to* $\Theta_{\vartheta_1\vartheta_2}$ *and let* $x = (r, \vartheta)$ *be an associated parabolic Morse minimizer for* (U, α)*. Then the corresponding* (ϑ, φ) *is a heteroclinic connection between the saddles*

$$(\vartheta^-, \vartheta^- + \pi) \text{ and } (\vartheta^+, \vartheta^+).$$

Moreover ϑ *is strictly increasing between* ϑ^- *and* ϑ^+*.*

Corollary

Let $\vartheta^- < \vartheta^+$ belong to $\Theta_{\vartheta_1\vartheta_2}$ and let $x = (r, \vartheta)$ be an associated parabolic Morse minimizer for (U, α). Then the corresponding (ϑ, φ) is a heteroclinic connection between the saddles

$$(\vartheta^-, \vartheta^- + \pi) \text{ and } (\vartheta^+, \vartheta^+).$$

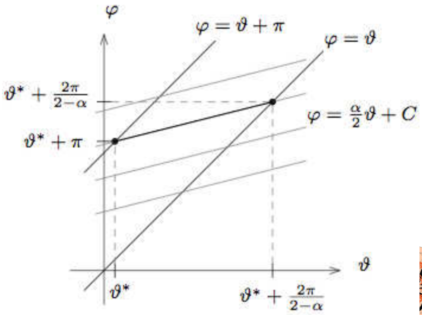
Moreover ϑ is strictly increasing between ϑ^- and ϑ^+.

In the isotropic case $(U \equiv 1)$ we have

$$\begin{cases} \vartheta' = 2\sin(\varphi - \vartheta) \\ \varphi' = \alpha \sin(\varphi - \vartheta). \end{cases}$$

Heteroclinic connections must satisfy

$$\vartheta^+ - \vartheta^- = \frac{2\pi}{2-\alpha}$$



Structural instability of Morse minimizing parabolic trajectories

- Parabolic trajectories for (DS) correspond to saddle-saddle heteroclinic connections for a planar dynamical system.

Structural instability of Morse minimizing parabolic trajectories

- Parabolic trajectories for (DS) correspond to saddle-saddle heteroclinic connections for a planar dynamical system.
- But generically the unstable manifold at a saddle falls into a sink, while the stable one emanates from a source, implying that parabolic trajectories do not exist.

Parabolic Trajectories as Phase Transition

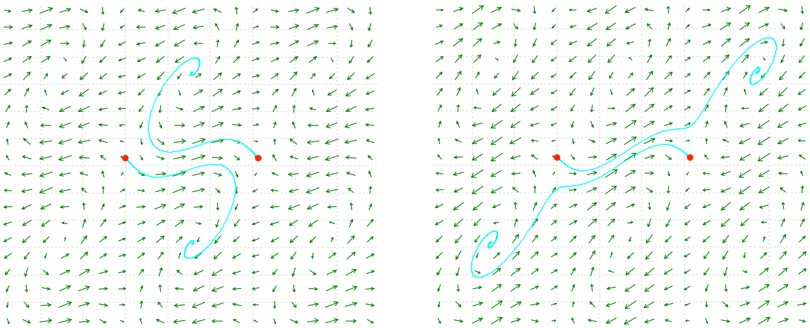


Figure: the two pictures represent the phase portrait of the planar dynamical system with $U(\vartheta) = 2 - \cos(2\vartheta)$, when $\alpha = 0.5$ (at left) or $\alpha = 1$ (at right). We focus our attention on the saddles $(0, \pi)$ and (π, π): from the mutual positions of the unstable manifold departing from $(0, \pi)$ and the stable one ending in (π, π) we deduce that the two vector fields are not topologically equivalent. By structural stability we infer the existence, for some $\bar{\alpha} \in (0.5, 1)$, of a saddle connection between $(0, \pi)$ and (π, π).

Introducing a Transition Parameter

We deal with the anisotropic Kepler problem in any dimension: more precisely V is such that:

- $V \in \mathcal{C}^2(\mathbb{R}^d \setminus \{0\})$, in particular $\mathcal{X} = \{0\}$;
- $V(x) = V(s)/r^\alpha$, $\alpha \in (0, 2)$;
- $V > 0$;
- V admits (at least) two non-degenerate and globally minimal c.c. $\xi^\pm$.

The previous discussion suggests to choose the homogeneity exponent $-\alpha$ as parameter. To clarify the role of the parameter, we let the potential vary in a class.

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The previous discussion suggests to choose the homogeneity exponent $-\alpha$ as parameter. To clarify the role of the parameter, we let the potential vary in a class.

For $\xi^+ \neq \xi^-$ in $\mathbb{S}^{d-1}$ and $V_{\min} > 0$, and let us define the metric spaces

$$\begin{aligned}\mathcal{U} &= \left\{ V \in \mathcal{C}^2(\mathbb{S}^{d-1}) : \begin{array}{l} s \in \mathbb{S}^{d-1} \text{ implies } V(s) \geq V(\xi^\pm) = V_{\min}; \\ \exists \delta > 0, \mu > 0 \text{ such that } |s - \xi^\pm| < \delta \\ \text{implies } V(s) - V(\xi^\pm) \geq \mu |s - \xi^\pm|^2 \end{array} \right\}, \\ \mathcal{V} &= \left\{ (V, \alpha) \in \mathcal{C}^2(\mathbb{S}^{d-1}) \times (0,2) : \ V \in \mathcal{U} \right\},\end{aligned}$$

the latter being equipped with the product distance.

Table of Contents

Inner and Outer Potentials

The property of a potential to admit parabolic minimizers is related to its behavior w.r.t. the fixed-endpoints problem

$$c(V) := \inf \left\{ \mathcal{A}\left([a,b];x\right) : a < b,\, x \in H^1(a,b),\, x(a) = \xi^-,\, x(b) = \xi^+ \right\}.$$

The value $c(V)$ is always achieved according to the following alternative.

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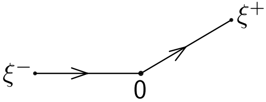
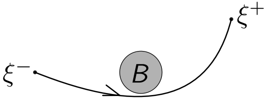
The value $c(V)$ is always achieved according to the following alternative.

In $:= \{V : c(V)$ is achieved by the juxtaposition of two homothetic motions, the first connecting ξ^- to the origin and the second the origin to $\xi^+\}$



Out $:= \{V : c(V)$ is achieved by motions which are uniformly bounded away from the origin $\}$



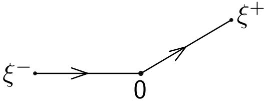
Inner and Outer Potentials

The property of a potential to admit parabolic minimizers is related to its behavior w.r.t. the fixed-endpoints problem

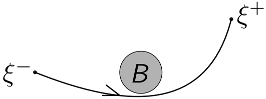
$$c(V) := \inf\left\{\mathcal{A}([a,b];x) : a<b,\ x\in H^1(a,b),\ x(a)=\xi^-,\ x(b)=\xi^+\right\}.$$

The value $c(V)$ is always achieved according to the following alternative.

$\mathrm{In} := \{V : c(V)$ is achieved by the juxtaposition of two homothetic motions, the first connecting ξ^- to the origin and the second the origin to $\xi^+\}$



$\mathrm{Out} := \{V : c(V)$ is achieved by motions which are uniformly bounded away from the origin $\}$



The sets In and Out enjoy the following properties:

- $\mathrm{In}\cap\mathrm{Out}=\emptyset$, $\mathrm{In}\cup\mathrm{Out}=\mathcal{V}$;
- In is closed;
- Out is open.

Structure Theorem

The role of the homogeneity parameter can be now clarified by the following property. Let $\Pi := \partial \mathrm{In} \cap \partial \mathrm{Out}$.

Lemma (Separation Property)

There exists an open nonempty set $\Sigma \subset \mathcal{U}$, and a continuous function $\bar{\alpha} : \Sigma \to (0,2)$ such that

$$\Pi = \{(V, \bar{\alpha}(V)) : \ V \in \Sigma\} \,.$$

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We can now characterize the set of potentials admitting parabolic Morse minimizers as the graph of the above function.

Main Theorem.

$V \in \mathcal{V}$ admits a parabolic Morse minimizer $\iff$ $V \in \Pi$.

Back to $\mathbb{R}^2$: Topological Constraints

Let $d = 2$, $U(\vartheta) := V(\cos\vartheta, \sin\vartheta)$. In this last part, for the sake of simplicity, let U be a positive, $\mathcal{C}^2$ Morse function such that every local minimum is indeed a global one.

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Since $\mathbb{R}^2 \setminus \{0\}$ is not simply connected, we can search for minimizers with respect to a given homotopy class:

connecting ξ^- and ξ^+
with $h \in \mathbb{Z}$ rotations around 0

$\Longleftrightarrow$

connecting $\vartheta^- := \arg\xi^-$,
$\vartheta^+ := \arg\xi^+ + 2h\pi$
in the universal covering

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Motivated by this, we introduce the set

$$\Theta := \{\vartheta \in \mathbb{R} : \vartheta \text{ is a (non-degenerate global) minimum for } u\}.$$

Parabolic Threshold and Collisionless Minimizers

Theorem

Let $\vartheta^-, \vartheta^+ \in \Theta$, $\vartheta^- \neq \vartheta^+$. Then there exists at most one $\bar{\alpha} = \bar{\alpha}(\vartheta^-, \vartheta^+) \in (0, 2)$ such that $V = (U, \alpha)$ admits a corresponding parabolic Morse minimizer associated with $(\vartheta^-, \vartheta^+, U)$ if and only if $\alpha = \bar{\alpha}$.

If such a $\bar{\alpha}$ does not exist, we define $\bar{\alpha}(\vartheta^-, \vartheta^+) := 0$.

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For any x_1, x_2 in the sector $(\vartheta^-, \vartheta^+)$, if $\alpha > \bar{\alpha}(\vartheta^-, \vartheta^+)$ then all fixed-time Bolza minimizers with endpoints x_1, x_2, within the sector, are collisionless.

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Theorem

For any $k \in \mathbb{Z} \setminus \{0\}$ and $T > 0$, if $\alpha > \bar\alpha(\vartheta^*, \vartheta^* + 2k\pi)$, for every minimum $\vartheta^* \in \Theta$, then there exists an action minimizing collisionless T–periodic trajectory winding k times around zero.

Table of Contents

Asymptotic solutions

We deal with the following class of trajectories having a prescribed asymptotic behaviour.

Definition

Given $T \in (0, +\infty]$, a *T-asymptotic solution* is a solution $\gamma \in \mathscr{C}^2((0,T), \widehat{\mathcal{X}})$ which pointwise solves the differential equation on $(0,T)$ and such that the following alternative holds:

(i) if $T < +\infty$, then $\gamma \in \mathscr{C}^0([0,T], \mathcal{X})$ and it experiences a *total collision* at the final instant $t = T$, namely

$$\lim_{t \to T^-} \gamma(t) = 0, \quad \textit{termed a time-}T\textit{ total collision trajectory;}$$

(ii) if $T = +\infty$, then $\gamma \in \mathscr{C}^0([0,+\infty), \mathcal{X})$ and

$$\lim_{t \to +\infty} \dot{\gamma}(t) = 0, \quad \textit{termed a completely parabolic trajectory.}$$

A Lagrangian version of Mc Gehee coordinates

Start with a Morse minimizer path $x = rs$, using polar coordinates where $r^2 = I = \|x\|_M^2$, $s = x/R$ and write the Lagrangian integral

$$\int_0^T \frac{1}{2}(\dot{r}^2 + r^2 \|\dot{s}\|_M^2) + \frac{U(s)}{r^\alpha}$$

Now, we change time and space variables:

$$\mathrm{d}t = r^{(2+\alpha)/2}\mathrm{d}\tau \ , \qquad (\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\tau} \ , \qquad \rho = r^{(2-\alpha)/4}.$$

Assume there is a total collision at $t = T$, then, by our asymptotic estimates:

$$\tau^* = \int_0^T \frac{\mathrm{d}t}{r^{(2+\alpha)/2}} = +\infty \ , \qquad T = \int_0^{+\infty} \rho^{2(2+\alpha)/(2-\alpha)}\mathrm{d}\tau$$

A new Lagrangian system

So we find a new constrained Lagrangian integral:

$$\int_0^{+\infty} \frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2 \left(\frac{1}{2}\|s'\|_M^2 + U(s)\right) .$$

Where the time limitation translates into a new integral constraint:

$$T = \int_0^{+\infty} \rho^{2(2+\alpha)/(2-\alpha)} \mathrm{d}\tau .$$

If we are dealing with free time minimizers, there is no such constraint.
In this case, the energy is zero.

A regularized Lagrangian system

The Euler-Lagrange equations for the new functional are

$$\begin{cases} -\left(\frac{4}{2-\alpha}\right)^2 \rho'' + \left(\|s'\|_M^2 + 2U(s)\right)\rho = 0 \\ -\left(\rho^2 s'\right)' + \rho^2 \nabla_s U(s) = \rho^2 \|s'\|_M^2\, s \end{cases}$$

and the null energy condition transforms into a new null energy condition:

$$\frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2\left(\|s'\|_M^2 - 2U(s)\right) = 0\,.$$

Mc Gehee coordinates and the monotonicity formula

Let $v = \frac{4}{2-\alpha}\frac{\rho'}{\rho}$, so that

$$\|s'\|_M^2 - 2U(s) = -v^2 = -\left(\frac{4}{2-\alpha}\right)^2\left(\frac{\rho'}{\rho}\right)^2 .$$

Then we have, as we my assume $\rho' < 0$,

$$(v^2)' = 4\frac{\rho'}{\rho}\|s'\|_M^2 = (2-\alpha)v\|s'\|_M^2 \implies v' = \frac{2-\alpha}{2}\|s'\|_M^2 < 0.$$

Hence we have a monotone quantity (the Sundman function). Moreover, we can eliminate ρ from the system:

$$\begin{cases} v' = \dfrac{2-\alpha}{2}\|s'\|_M^2 \\ -\dfrac{2-\alpha}{2}vs' - s'' + \nabla_s U(s) = \|s'\|_M^2 s . \end{cases}$$

Central configuration as stationary solutions to the subsystem

The subsystem has stationary solutions corresponding to central configurations:

$$\begin{cases} v' = 0\ , \qquad v^2 = 2U(s_0) \\ \nabla_s U(s_0) = 0\ . \end{cases}$$

When they are nondegenerate, such stationary solutions are (linearly) hyperbolic if and only if

$$\nabla_s^2 U(s) > -\frac{(2-\alpha)^2}{8} U(s_0)$$

A new metric and a new potential

Let us consider a new variable $y(\tau) = \rho(\tau)s(\tau)$. The regularized Lagrangian has a new metric and a new potential. The metric has been deformed in the radial direction:

$$c_\alpha = \left(\frac{4}{2-\alpha}\right)^2 - 1 \, .$$

Then, the new metric has coefficients

$$g_{ij}(y) = \delta_{ij} + c_\alpha \frac{y_i y_j}{\rho^2}$$

The potential also has been changed into $\tilde{U}(y) = \rho^2 U(s)$: it is now homogeneous of degree two.

The asymptotic analysis of an a.s.

Asymptotic estimates (potential, U)

(ii) $\lim_{t \to T^-} U(\gamma(t)/|\gamma(t)|) = b$

Asymptotic estimates (angular part, $s(t) = \gamma(t)/|\gamma(t)|$)

(iii) $\lim_{t \to T^-} \mathrm{dist}(\mathcal{C}^b, s(\tau)) = 0$, where $\mathcal{C}^b := \{\text{c.c. for } U \text{ at level } b\}$.

Definition

Given a central configuration s_0 for the potential $\mathcal{U}$ and an asymptotic solution γ, we say that γ is an s_0-asymptotic solution (s_0-a.s.) if

$$\lim_{t \to T^-} s(t) = s_0.$$

Consequences of the asymptotic analysis

- The set of s_0-a.s. is not empty, it indeed contains the s_0-homothetic motions

  $$\gamma_0(t) = r_0(t) s_0.$$

  where s_0 is a c.c. and r_0 solves the 1-dim. α-Kepler problem
- When $h = 0$ we find the s_0-homothetic parabolic motions

  $$\gamma^*(t) = [K\beta(t)]^{\frac{2}{2+\alpha}} s_0$$

  (expanding or collapsing).

Asympthotic estimates imply that any s_0-a.s. can be understood as a "perturbation" of an s_0-homothetic parabolic solution

Towards a Morse Index for s_0-a.s.

- s_0-a.s. are critical points for $\mathcal{A}$ in the following sense:

Lemma

Let γ be an s_0-a.s.. Then γ is a Gateaux $\mathscr{C}_0^\infty$-critical point for $\mathcal{A}$, that is

$$d\mathcal{A}(\gamma)[\xi] = \int_{\mathrm{supp}\xi} \langle \dot\gamma, \dot\xi \rangle_M + \int_{\mathrm{supp}\xi} \langle \nabla U(\gamma), \xi \rangle_M = 0,$$

for any $\xi \in \mathscr{C}_0^\infty(0, T; \mathcal{X})$, T finite or not.

We compute the quadratic form associated to $d^2\mathcal{A}(\gamma)$ in a new set of variables (named after McGehee). These coordinates blows-up the collision singularity by a scaling in time and make undiscernible collision motions and parabolic ones

Table of Contents

Variational version of McGehee coordinates

Let γ be an s_0-a.s., $\gamma(t) = r(t)s(t)$.

(1) Inspired by the behaviour of $r(t)$ as $t \to T$, $r(t) \sim [K\beta(t)]^{2/(2+\alpha)}$ on asymptotic motions, we rescale the time variable

$$\tau = \tau(t) := \int_0^t r^{-\frac{2+\alpha}{2}}(\xi) d\xi$$

obtaining a monotone increasing $\tau \in [0, +\infty)$, indeed

$$\lim_{t \to T^-} \tau(t) = \lim_{t \to T^-} \int_0^t \frac{d\xi}{\beta(\xi)} = +\infty$$

collisions are moved to $+\infty$, any a.s. is now defined on $[0, +\infty)$

Variational version of McGehee coordinates

(2) We now rescale the radial variable defining

$$\rho(\tau) = r^{\frac{2-\alpha}{4}}(t(\tau)) \in [0, +\infty)$$

which has the asymptotic behavior

$$\lim_{\tau\to+\infty} \frac{\rho'(\tau)}{\rho(\tau)} = \delta_\alpha = \begin{cases} -\dfrac{2-\alpha}{4}\sqrt{2b}, & \text{if } T < +\infty \\ +\dfrac{2-\alpha}{4}\sqrt{2b}, & \text{if } T = +\infty \end{cases}$$

ρ has an exponential decay/growth, with rate δ_α

- if $\rho_0(\tau)$ corresponds to the radial variable of a s_0-homothetic parabolic solution, γ^*, then

$$\frac{\rho_0'(\tau)}{\rho_0(\tau)} = \delta_\alpha$$

The new functional $\mathbb{J}$

Lemma

Let $\gamma(t) = r(t)s(t)$ be a s_0-a.s. and let $y(\tau) = \rho(\tau)s(\tau)$ be the corresponding trajectory in the McGehee variables.
Then y is a Gateaux $\mathscr{C}_0^\infty$-critical point for

$$\mathbb{J}(y) := \int_0^{+\infty} \frac{c_\alpha}{2}(\|y\|'_M)^2 + \frac{1}{2}\left[\|y'\|_M^2 + 2\mathcal{U}(s_0)\,\|y\|_M^2\right] + W(y) + h(\|y\|_M^\beta - T)$$

where $W(y) := \|y\|_M^2\,[\mathcal{U}(s) - \mathcal{U}(s_0)]$, $\beta > 2$, and $c_\alpha > 3$.

- When $T = +\infty$ then $h = 0$: the last term appears only when $T < +\infty$. In this case the time scaling imposes the constraint

$$\|\rho^\beta\|_{L^1([0,\infty))} = T.$$

Table of Contents

The Hessian of $\mathbb{J}$ on s_0-a.s.

- Let $y(\tau) = \rho(\tau)s(\tau)$ be an s_0-a.s. then

$$d^2\mathbb{J}(y)[\eta,\xi] = \int_0^{+\infty} \Big\langle [R(\tau) + D^2U(y)]\eta, \xi \Big\rangle_M + \langle Q(\tau)\eta', \xi \rangle_M + \\ + \Big\langle Q^T(\tau)\eta, \xi' \Big\rangle_M + \langle P(\tau)\eta', \xi' \rangle_M, \quad \eta, \xi \in W_0^{1,2}([0,+\infty), \mathcal{X})$$

where $R(\tau)$, $Q(\tau)$ and $P(\tau)$ depend on $\rho(\tau)$, $\rho'(\tau)$, $s(\tau)$ and s_0.

- By means of the asymphthotic estimates, we define the limit matrices

$$R_0 := \lim_{\tau\to+\infty} R(\tau) \qquad Q_0 := \lim_{\tau\to+\infty} Q(\tau) \qquad P_0 := \lim_{\tau\to+\infty} P(\tau)$$

- P_0 and R_0 are positive definite

The coefficients of the Hessian for an s_0-asymptotic solution

We denote by $u \otimes_M v$ the tensor product of the two vectors u and v made with respect to the mass scalar product, i.e. for any vector η, we obtain $(u \otimes_M v)\eta = \langle v, \eta \rangle_M u$. We also observe that $(u \otimes_M v)^T = (v \otimes_M u)$. Given an s_0-asymptotic solution y, let us introduce the endomorphisms

$$
\begin{aligned}
P(\tau) &= c_\alpha s \otimes_M s + I \\
Q(\tau) &= c_\alpha \left(\frac{\rho'}{\rho}\right) (I - s \otimes_M s) + c_\alpha s \otimes_M s', \\
R(\tau) &= \left[c_\alpha \left(\frac{\rho'}{\rho}\right)^2 + \beta(\beta-2) h \rho^{\beta-2} \right] s \otimes_M s \\
&\quad + \left[2U(s_0) - c_\alpha \left(\frac{\rho'}{\rho}\right)^2 + \beta h \rho^{\beta-2} \right] I \\
&\quad + c_\alpha s' \otimes_M s' - c_\alpha \left(\frac{\rho'}{\rho}\right) (s \otimes_M s' + s' \otimes_M s) .
\end{aligned}
$$

The coefficients of the Hessian for an homothetic solutions

When the motion is homothetic at zero energy, the parameters reduce to

$$\begin{aligned}
P_0 &= c_\alpha s_0 \otimes_M s_0 + I, \\
Q_0 &= c_\alpha \bar{\delta}_\alpha (I - s_0 \otimes_M s_0) \\
R_0 &= c_\alpha \bar{\delta}_\alpha^2 s_0 \otimes_M s_0 + \left(2U(s_0) - c_\alpha \bar{\delta}_\alpha^2\right) I \\
&= c_\alpha \bar{\delta}_\alpha^2 s_0 \otimes_M s_0 + \frac{(2-\alpha)^2}{8} U(s_0) I.
\end{aligned}$$

The Hessian of $\mathbb{J}$ on s_0-homothetic parabolic motions

- It turns out that when $y^*(\tau) = \rho_0(\tau)s_0$ is a s_0-homothetic parabolic motion the bilinear form reduces to

$$d^2\mathbb{J}(y^*)[\eta,\xi] = \int_0^{+\infty} \left\langle [R_0 + D^2\widetilde{U}(s_0)]\eta, \xi \right\rangle_M + \langle P_0\eta', \xi' \rangle_M$$

where $\widetilde{\mathcal{U}}(x) = |x|_M^2\mathcal{U}(x)$, $x \in \mathcal{X}$.

- We introduce the following spectral condition

[BS]: the matrix $R_0 + D^2\widetilde{\mathcal{U}}(s_0)$ is positive definite

Let $\mu_1 :=$ the smallest eigenvalue of $D^2\widetilde{\mathcal{U}}(s_0)$, then

$$[\mathrm{BS}] \;\Leftrightarrow\; \mu_1 > -\frac{(2-\alpha)^2}{8}\mathcal{U}(s_0)$$

Finiteness of the Morse index

- Fixed an s_0-a.s. y we define the quadratic form on $W_0^{1,2}([0,+\infty),\mathcal{X})$

  $$\mathcal{Q}(\eta) := d^2\mathbb{J}(y)[\eta,\eta]$$

  and the *Morse index of* y as

  $$i_{Morse}(y) = \dim E_-(\mathcal{Q}) \in \mathbb{N}\cup\{+\infty\}$$

  where $E_-(\mathcal{Q})$ is the negative spectral space of a realisation of $\mathcal{Q}$.

When $T=+\infty$ the Morse index of y coincides with the Morse index of the corresponding a.s. γ (seen as a critical point on $\mathcal{A}$). When $T<+\infty$, the difference between these two indices is at most one: this is due to the presence of the 1-codimensional T-constraint

Finiteness of the Morse Index

Lemma

Let y^* be an s_0-homothetic parabolic solution, then

- $\mathcal{Q}$ is an equivalent norm on $W_0^{1,2}([0,+\infty);\mathcal{X}) \quad \Longleftrightarrow \quad$ [BS] holds
- $i_{Morse}(y^*)$ is finite (and $=0$) $\quad \Longleftrightarrow \quad$ $[\mathrm{BS}]_=$ holds

Theorem [B.-Secchi (2008), B.-Hu-Portaluri-Terracini (2017)]

Let y be an s_0-a.s., then

- $[\mathrm{BS}] \quad \Longrightarrow \quad i_{Morse}(y) < +\infty$
- $\neg[\mathrm{BS}]_= \quad \Longrightarrow \quad i_{Morse}(y) = +\infty$

Proof

$\mathcal{Q}(\eta) \geq \dfrac{\varepsilon}{2}\|\eta\|^2_{W_0^{1,2}([0,+\infty))} - k\|\eta\|^2_{L^2(0,T)}$ for some $\varepsilon, k > 0$. q.e.d.

We infer that not only s_0-a.s. to a minimal central configuration but also s_0-a.s. to a saddle one with a “not so negative minimal eigenvalue” has finite Morse index. Hence these asymptotic motions may be responsible of a change in the topology of the sub-levels of the Lagrangian action and actually plays a role in the topological balance.

Examples ($\alpha = 1$)

- The Lagrange c.c. (regular triangle) satisfies [BS] for any m_1, m_2, m_3,
- collinear c.c. 3 equal masses does not satisfy [BS],
- the regular N-gon with $N \geq 4$ does not satisfy [BS].

Further properties related with BS: Fredholmness

The Fredholm property plays a crucial role both in bifurcation and in index theories

Theorem

Let y be an s_0-a.s. then $\mathcal{Q}$ is Fredholm on $W_0^{1,2}([0,+\infty);\mathcal{X}) \iff$ [BS] holds

Table of Contents

The linearized hamiltonian system

- Let s_0 be a c.c. satisfying [BS] and y be an s_0-a.s.
- We consider the Hamiltonian system (naturally associated to $\mathcal{Q}$)

$$z'(\tau) = H(\tau)\, z(\tau) \qquad (H)$$

where $H(\tau) := \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix} B(\tau)$

and

$$B(\tau) := \begin{bmatrix} P^{-1}(\tau) & -P^{-1}(\tau)Q(\tau) \\ -Q^{\mathsf{T}}(\tau)P^{-1}(\tau) & Q^{\mathsf{T}}(\tau)P^{-1}(\tau)Q(\tau) - \widetilde{R}(\tau) \end{bmatrix}.$$

Intersection of Lagrangians: the geometrical index

- For any $\tau_0 \in [0, +\infty)$, let $\psi_{\tau_0} : [0, +\infty) \to \mathrm{Sp}(2N)$ be the matrix-valued solution of the Hamiltonian system (H) such that $\psi_{\tau_0}(\tau_0) = I$. We define the stable space as

$$E^s(\tau_0) = \{v \in \mathbb{R}^{2N} | \lim_{\tau \to +\infty} \psi_{\tau_0}(\tau)\, v = 0\}$$

- Let $L_0 := \mathbb{R}^N \times \{0\}$ be the (horizontal) Dirichlet Lagrangian.

Definition

We define the geometrical index of y as the integer given by

$$i_{\mathrm{geo}}(y) = -\mu\big(E^s(\tau_0), L_0; \tau_0 \in [0, +\infty)\big),$$

where the integer μ is the Maslov index for ordered pairs of paths of Lagrangian subspaces.

The Index Theorem and ...

Index Theorem `[B.-Hu-Portaluri-Terracini (2017)]`

Let s_0 be a c.c. satisfying [BS]. Given an s_0-a.s. y there holds

$$i_{Morse}(y) = i_{geo}(y).$$

Future projects

- Explicit computations of the index for some trajectories;
- the Maslov index is crucial in the investigation of the linear stability or instability of a critical points and an Index Theorem is the fundamental device in order to relate the linear stability properties to the Morse index of a solution (if a relation really exists...);
- in bifurcation theory its non-triviality is a sufficient condition to detect bifurcation of critical points from the trivial branch.