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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas **Email/Phone:** rmvargas@ciencias.unam.mx

Speaker's Name: Nancy Hingston

Talk Title: Loop Products and Self-intersection

Date: 08 / 17 / 18 **Time:** 9 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk, Nancy Hingston presented a joint work with Nathalie Wahl. She presented simplified, chain-level definitions for the Chas-Sullivan "loop" product and coproduct on the homology of the space of maps of the circle (LM) into a compact oriented manifold (M). She also discused interactions between geometry and the loop coproduct when the homology class (X) on LM has a representative with no self-intersections of order larger than K, the the K-fold coproduct of X is trivial

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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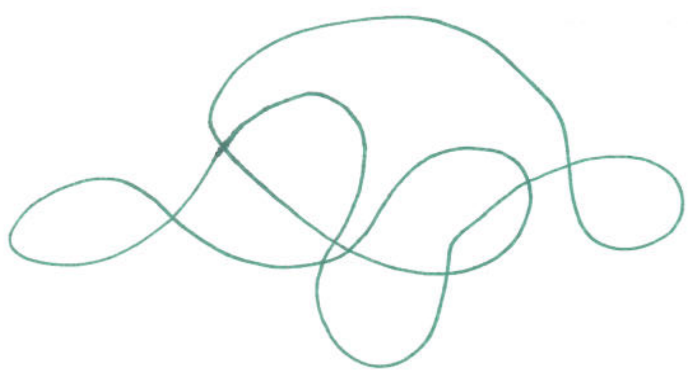
Loop Products and Self-Intersections

Motivation and Intuition



MSRI Aug 17, 2018

Nancy Hingston
The College of New Jersey

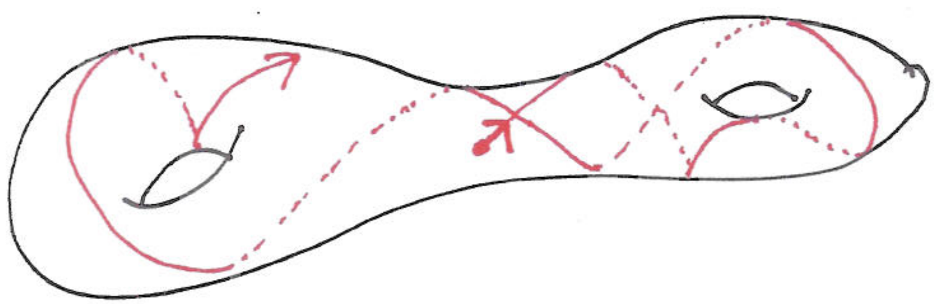
Recent joint work with Nathalie Wahl on String Topology

First : Geometric Motivation

Relationship between Products and geodesics

geometry :

M compact, oriented Riemanian manifold dim n



Seach for closed geodesics on M.

Poincaré, Birkhoff, Morse, ...

We know (Assume $\pi_1 M = 0$)

Gromoll-Meyer (69), Sullivan-Vigué Poirrier (77):

For most manifolds, every metric on M has infinitely many closed geodesics.

Rademacher: For most metrics on any manifold, there are infinitely many closed geodesics

Birkhoff-Lusternik Schnirelmann, Bangert, Franks, Grayson: there are infinitely many closed geodesics for every metric on S^2.

BUT ...

- Grove '73 Isometries on spheres with finitely many invariant geodesics
- Katok '73 Ziller '77 Nondegenerate Finsler metrics with finitely many closed geodesics
- $M = S^n$, $n \geq 2$ $M = \mathbb{C}P^k$, $k > 1$

  $\min \# \geq 1$

Morse Theory.

Geometry

geodesics

index growth

Topology

Loop space

String products

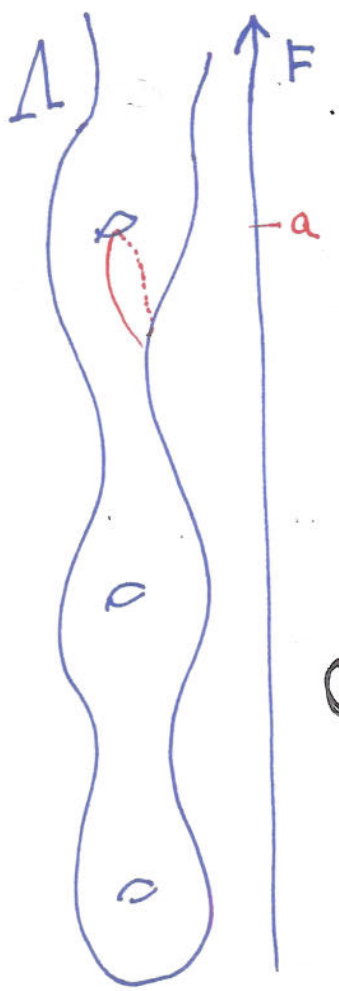
Method (Morse, ...) Fix a metric on M.

$\Lambda M = \{ \gamma : S^1 \to M \}$ Free Loop Space

$E : \Lambda M \to \mathbb{R}$ $\quad E(\gamma) = \int_{S^1} |\dot{\gamma}|^2 dt$

Better function $F = \sqrt{E} \approx$ length

Critical points $\equiv$ Closed geodesics on M



$H_k(\Lambda) \simeq$ Critical points index k

Given $X \in H_*(\Lambda)$

$Cr(X) \equiv \inf \{ a \in \mathbb{R} \mid X \text{ supported on } \Lambda^{\leq a} \}$

is a critical value of F

Q Can we use this correspondence to "count" closed geodesics on M?

Difficulty: Iterates

Inside the free Loop space ΛM,

<u>one</u> closed geodesic γ appears as an infinite

sequence $\gamma, \gamma^2, \gamma^3, \ldots$ different $\left\{ \begin{array}{l} \text{length} \\ \text{index.} \end{array} \right.$

$$H_k(\Lambda) \approx \underset{\text{index } k}{\text{Critical points}}$$

Q. Is there an algebraic structure on $H_*(\Lambda)$ (e.g. product) that corresponds to iteration of closed geodesics?

A. In many critical cases, <u>YES</u>.

PRODUCTS ON LOOP SPACES

(topology)

Based Loop Space $\Omega M = \{ \gamma \in \Lambda M \mid \gamma(0) = * \}$

$* \in M$

$\Longrightarrow$ Pontryagin Product on $H_*(\Omega M)$

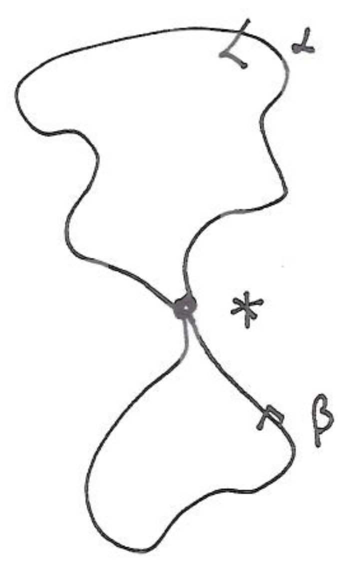
$A, B \subseteq \Omega$ cycles

$[A] \in H_i(\Omega)$

$[B] \in H_j(\Omega)$

$$[A] \bullet_{PP} [B] = \left[\left\{ \alpha \cdot \beta \mid \alpha \in A, \beta \in B \right\} \right] \in H_{i+j}(\Omega)$$

concatenation



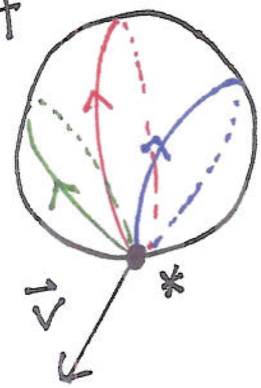
Example: $M = S^n$, $n > 1$

Some cycles in ΩM:

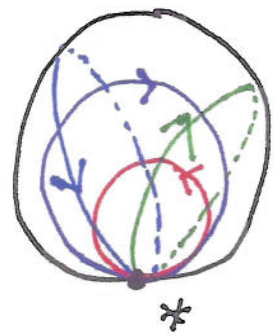
$U^0 = \{\text{constant loop at } *\}$

$A^{n-1} = \left\{ \begin{array}{l} \text{Circles beginning at } * \in M \\ \text{with tangent vector } \vec{V} \end{array} \right\}$

A



B



$B^{2n-2} = \{\text{Circles beginning at } *\}$

$[U] \bullet_{pp} [A] = \underline{[A]}$

$[A] \bullet_{pp} [A] = \underline{[B]}$

in standard metric

$Cr\,[A] = \underline{2\pi}$

$Cr\,[A] \bullet_{pp} [A] = \underline{2\pi}$

$Cr\,[A]^{2m} = \underline{\quad}$

$\shortparallel$

$Cr\,[A]^{2m-1} = \underline{2m\pi}$

$[A]$ is non nilpotent:

$[A]^m \neq 0$

$H_*(\Omega S^n) = \mathbb{Z}[A]$

Pontryagin ring

$\Longrightarrow$ Coproduct $\vee$ on $H_*(\Omega)$ (Sullivan, Goresky–H)

$A \subseteq \Omega M \qquad [A] \in H_k(\Omega)$

cycle

$A \times I \subseteq \Omega \times I \supseteq \widehat{\mathcal{F}}_\Omega = \{(\gamma, t) \mid \gamma(0) = \gamma(t)\}$ 

$$A \times I \cap \widehat{\mathcal{F}}_\Omega \subseteq \widehat{\mathcal{F}}_\Omega \xrightarrow{\text{cut}} \Omega \times \Omega$$

$$(\gamma, t) \longmapsto \left(\gamma\big|_{[0,t]}, \gamma\big|_{[t,1]}\right)$$

$$H_k(\Omega) \to H_{k+1-n}(\Omega \times \Omega) \cong \bigoplus_{i+j=k+1-n} H_i(\Omega) \otimes H_j(\Omega)$$

$$[A] \qquad\qquad \vee [A]$$

"Find all self-intersections and cut the loops apart."

... Well not quite ...

- Definition of coproduct is not rigorous

- Output in $H_*(\Omega \times \Omega, \Omega \times \{*\} \cup \{*\} \times \Omega)$

Products on ΛM

$\Longrightarrow$ Chas-Sullivan 1999 $A, B \subseteq \Lambda$ cycles "Assume transversality"

$$[A] \bullet_{cs} [B] = \left[\left\{ \alpha \cdot \beta \;\middle|\; \begin{array}{l} \alpha \in A, \beta \in B \\ \underline{\text{and}}\ \alpha(0) = \beta(0) \end{array} \right\} \right]$$

$$H_i(\Lambda) \otimes H_j(\Lambda) \longrightarrow H_{i+j-n}(\Lambda)$$

$\longrightarrow$ Coproduct on $H_*(\Lambda)$:

$$\vee_\Lambda : H_k(\Lambda) \to H_{k+1-n}(\Lambda \times \Lambda, \Lambda \times \Lambda^\circ \cup \Lambda^\circ \times \Lambda)$$

Λ° = trivial loops

Idea: Find all self-intersections, and cut.

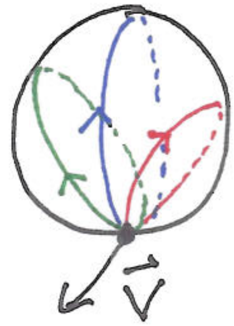
Better definitions: Cohen-Jones Goresky-H
H-Wahl

Get rid of boundary terms H-Wahl

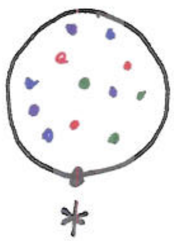
Example $M = S^n$

Some cycles in ΛM:

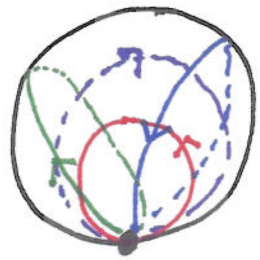
U^0 = the constant loop at $*$



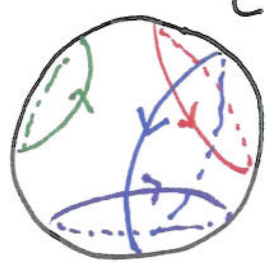
A^{n-1} = Circles beginning at $*$ with tangent vector $\vec{V}$



E^n = constant loops on S^n



B^{2n-2} = Circles beginning at $*$



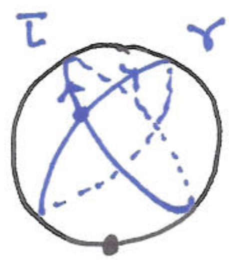
C^{3n-2} = All circles, great and small

$[C] \bullet_{cs} [E] = \underline{[C]}$

$[A] \bullet_{cs} [A] = \underline{0}$

$[C] \bullet_{cs} [C] = \left[\left\{ \gamma \cdot \tau \;\middle|\; \gamma \in C,\ \tau \in C \text{ \underline{and} } \gamma(0) = \tau(0) \right\} \right]$

$[C]^m \neq 0$



n even $\Rightarrow H_*(\Lambda S^n; \mathbb{Q})$ generated by U, E, A, C

(computed by Cohen-Jones-Yau)

(algebra)

Coproduct on homology $\longleftrightarrow$ Product on cohomology

$$\langle \vee A, x \otimes y \rangle = \langle A, x \cdot y \rangle$$

$\uparrow$ coproduct on homology

$\uparrow$ GH product on cohomology

if $A \in H_k(\Lambda)$

$x \in H^i(\Lambda), y \in H^j(\Lambda)$

$i+j = k+1-n$

Geometry, Iteration and Products

Examples of $\begin{cases} \text{Poincaré Duality in } \Lambda, \Omega \\ \text{Relations b/n products and geometry} \end{cases}$

- Basic Inequalities

$$X, Y \in H_*(\Lambda M) \Rightarrow Cr(X \cdot Y) \leq Cr(X) + Cr(Y)$$

$$x, y \in H^*(\Lambda M) \Rightarrow Cr(x \cdot y) \geq cr(x) + Cr(y)$$

- Old theorems Restated

Bott (56) Suppose all closed geodesics on M are nondegenerate. Then

$$\forall X \in H_*(\Lambda),\ Cr(X^m) < m\,Cr(X) \text{ for large } m$$

$$\forall x \in H^*(\Lambda)\ \ Cr(x^m) > m\,Cr(x) \text{ for large } m$$

H. (93) If $\exists\, X \in H_*(\Lambda) : Cr(X^m) = m\,Cr(X)\ \forall m$

$\Rightarrow$ M has infinitely many closed geodesics

(97) If $\exists\, x \in H^*(\Lambda) : Cr(x^m) = m\,Cr(x)\ \forall m$

$\Rightarrow$ M has infinitely many closed geodesics

- Index Growth

Bott (56) γ closed geodesic on M.

"Formula for the index of the iterates γ^m:"

$$m \cdot \mathrm{Index}(\gamma) - (m-1)(n-1) \leq \mathrm{Index}(\gamma^m)$$

$$\leq m \cdot \mathrm{Index}(\gamma) + (m-1)(n-1)$$

4 inequalities

4 products

Minimal growth
$\Rightarrow$ nontrivial CS product

Maximal growth
$\Rightarrow$ nontrivial Loop cohomology product

Equality $\Longleftrightarrow$ Nontrivial Products

Ex spheres, projective spaces

All geodesics closed

$\Rightarrow$ index growth minimal and maximal

Products highly nontrivial

Connections in Floer theory

Viterbo, Salamon-Weber,

Abbondandolo-Schwarz,

Cohen-Hess-Voronov:

$H_*(\Lambda M, \bullet_{cs}) \cong HF_*(T^*M, \text{pair-of-pants product})$

Ginzburg-Gürel

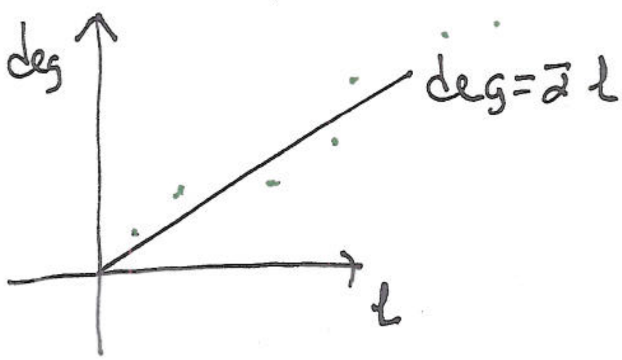
Local Floer homology

Infinitely many periodic orbits when the pair-of-pants product is level-nonnilpotent.

(Conley conjecture)

- Resonance [Rademacher, H] ('13)

Fix a metric on S^n. The points $(cr(x), \deg(x))$ lie at bounded distance from a line



of slope $\bar{\alpha}$ $\qquad \dfrac{\#\,\text{conj. pts}}{\text{length}}$

Proof uses homology <u>and</u> cohomology products.

(topology)

Recent work on loop products
aka String topology
w/ Nathalie Wahl (Ralph Cohen)

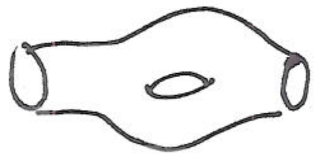
(1) New definitions $\begin{cases} \text{Chas-Sullivan product } \wedge \\ \text{Loop coproduct } \vee \end{cases}$

(2) "Lift" of the coproduct

$$H_*(\Lambda) \longrightarrow H_*(\Lambda) \otimes H_*(\Lambda)$$

(3) "Mixed Products"

(3) The (signed) incestuous product is trivial:

 $\equiv 0$ ()

coproduct product

If n is odd, $\bullet(V) \equiv 0 \pmod{\Lambda^0}$

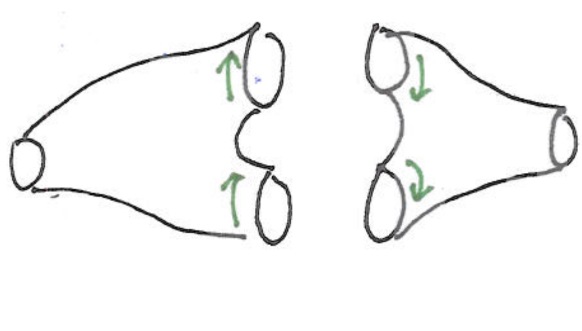
If n is even, $\bullet(V) \equiv 0 \left(\begin{array}{c}\bmod 2 \\ \bmod \Lambda^0\end{array}\right)$

However e.g. $M = S^3$ $k \geq 14$ even

$\frac{k}{2} \equiv 3 \ (4)$

$\Rightarrow \exists\, X \in H_k(\Lambda S^3)$:

$\bullet(\Delta_1, \Delta_2) \vee X \neq 0$

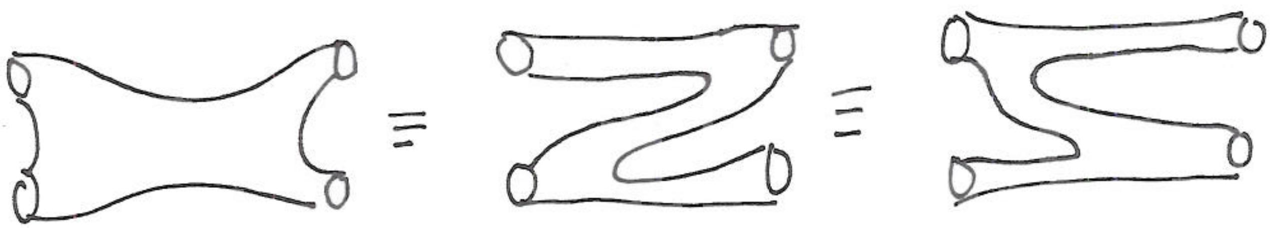


(3) ~~[illegible]~~

What about $V(\cdot)$?

For a field theory we would have

Frobenius: $V(A\cdot B) = (VA)\cdot B = A\cdot(VB)$



This seems to be true only for

trivial coproducts.

Sullivan formula

$$V(A\cdot B) = (VA\cdot B) + A\cdot (VB)$$

Sayp: The self-intersections in $A\cdot B$ come from the self-intersections in A and the self-intersections of B.

— turns out to be FALSE. Examples

$$\text{LHS} \neq \text{RHS}$$

Geometric Interpretation of LHS-RHS in the finite-dimensional approximation to ΛM:

$$V(A\cdot B) - \big((VA\cdot B) + A\cdot (VB)\big)$$

is picking up 2^{nd} order intersections of A, B.

(4) Support

- CS product $[A] \bullet_{cs} [B]$ $A, B \subseteq \Lambda$

  If rep's A, B have disjoint base points,

  i.e. $\{\gamma(0)\}_{\gamma \in A} \cap \{\gamma(0)\}_{\gamma \in B} = \phi$

  then $[A] \bullet_{cs} [B] = 0.$ ("obvious")

- Coproduct VA $A \subseteq \Lambda$

  If rep A has no loops with self-intersections,

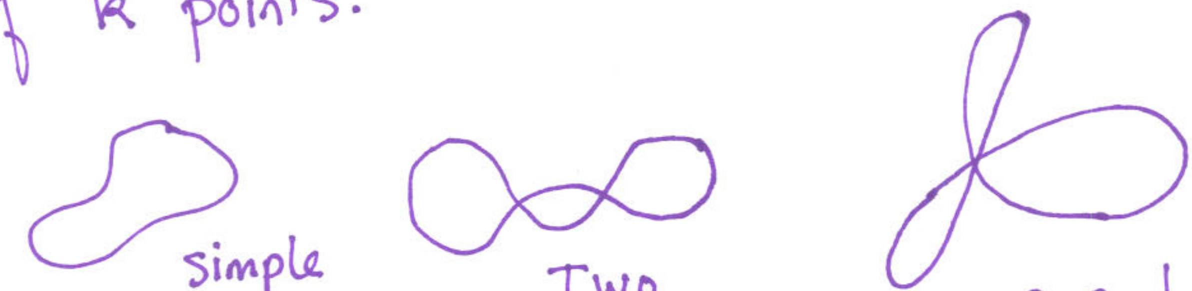
  then $[VA] = 0.$

  Should be true!

  Proof not available until now.

  Use new definitions!

<u>Def</u>: A loop $\gamma \in \Lambda$ has a <u>k-fold intersection</u> at $p \in M$ if $\gamma^{-1}\{p\}$ consists of k points.



simple loop. (1-fold intersections)

Two 2-fold intersections

a 3-fold intersection.

<u>Theorem</u> If $Z \subset \Lambda$ is a cycle & every nonconstant loop in the image of Z has at most k-fold intersections, then

(i) $\hat{V}^k[Z] = 0$.

(ii) If $[x_1], \dots, [x_k] \in H^*(\Lambda)$, then $\langle [x_1] \circledast [x_2] \circledast \cdots \circledast [x_k], [Z] \rangle = 0$.

<u>Sharp</u> for spheres, projective spaces!

Intuition:

The coproduct V looks for self-intersections, and cuts them apart.

No self-intersections $\Rightarrow V[Z] = 0$ ✓

Ex. $\hat{V}\left[\!\left[C^{3n-2} \right]\!\right] = 0$ (trivial loops allowed!)

$\uparrow$ all circles great & small

The coproduct is not foolish enough to mistake the tautology $\gamma(0) = \gamma(0)$

(Set $\gamma(0) = \gamma(s)$, let $s \to 0$)

For a self-intersection at the base point!