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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas **Email/Phone:** rmvargas@ciencias.unam.mx / 5104243513

Speaker's Name: Eva Miranda

Talk Title: Desingularizing singular symplectic structures to understand their associated dynamics

Date: 08 / 17 / 18 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk E. Miranda presented a joint work with Victor Guillemin and Jonathan Weitsmann, she presented a desingularization technique for symplectic (and contact) structures with singularities which appear modeling some problems in Celestial Mechanics and describe several applications to the study of their Hamiltonian (and Reeb) Dynamics. She also showed that those singular symplectic structures can be formalized as smooth Poisson structures which are symplectic away from a hypersurface and satisfy some transversality properties.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

b-Poisson structures in the universe of Poisson Structures Assuming transversality and that the critical set is no empty this is the set of Poisson structure:

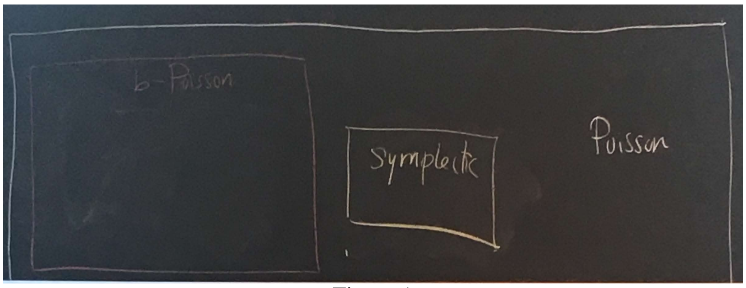

Figure 1

Definition 1. Symplectic structure

$\omega \in \Omega^2(M),\ \omega -$ symplectic structure and $d\omega = 0$. So, ω is a closed 2-form and non degenerate.

b-vector field is a Poisson structure

- $\Pi \in \Lambda^2(TM)$
- $\{f,g\} = \Pi(df,dg)$
- $\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \rightsquigarrow \{f,g\}$ standard Poisson bracket

Review of General definition of a Poisson structure.

$$\{,\} : C^\infty(M) \times C^\infty(M) \to K^\infty(M),$$

1 $\mathbb{R}$- bilinear, antysimetric satisfying

2 Leibniz identity:$\{f,gh\} = \{f,g\}h + g\{f,h\}$

3 Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Jacobi condition in b-vector fields can be summarized by:

Jacobi if and only if $[\Pi,\Pi] = 0$ (Schouten-Nijenhuis Bracket)

Schouten-Nijenhuis bracket is the extension of a Lie bracket from vector fields to multi-vector fields.

Example of a b-Poisson structure

Going to dimension two. Take a sphere and, the field h (see Figure 2).
Take the following Poisson structure:

- $\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ this is a b-vector field.
- $[\Pi,\Pi] = 0$. In dimension 2 this is always satisfied.

In this case,

1. $Z = \{h = 0\}$. The equator is a critical set.
2. $\Pi_{|_Z} = 0$. In this case I will get the Zero Poisson structure.

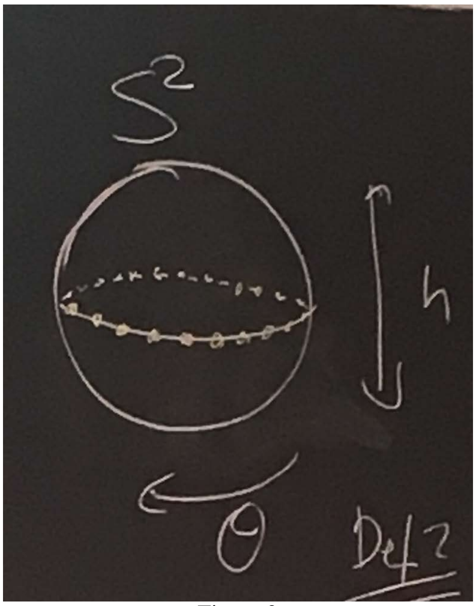

Figure 2

Definition 2. Hamiltonian vector field

If I have a Poisson structure very easily we can associate to it a Hamiltonian vector field of f by defining:

$$X_f = \Pi(\mathrm{df}, \cdot)$$

This vector field is called the Hamiltonian vector field of f.

Definition 3 Consider $\mathscr{D} = \left\{X_f, f \in C^\infty(M)\right\}$, it turns out that

$\mathscr{D}$ is integrable in the sense of Frobenius theorem so, I have a foliation, $\mathscr{F}$, and this foliation is called the symplectic foliation.

Example of b-Poisson structure in higher dimensions Consider a Poisson manifold (N^{2n-1}, Π). Consider a Codimension 1 Foliation $\mathscr{F}$. $\mathscr{F}$ Symplectic foliation. Assume that there exist a vector field X that is transverse to this symplectic foliation $\mathscr{F}$ and which is a Poisson vector field.

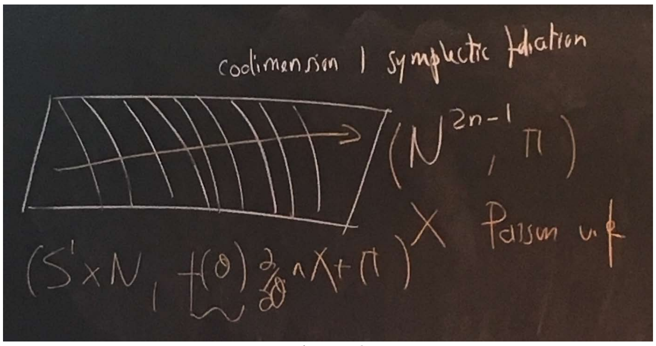

Figure 3

Then $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \Pi))$ with function f vanishes linearly.

Desingularizing singular symplectic structures to understand their associated dynamics

Connections for Women: Hamiltonian Systems, from topology to applications through analysis

Eva Miranda (UPC & Observatoire de Paris)

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Outline

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

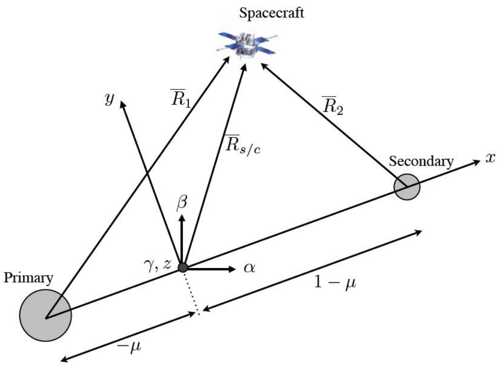


Figure: Circular 3-body problem

Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbf{R}^2 \times \mathbf{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+$, can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

which extends to a b^3-symplectic structure on $\mathbf{R} \times \mathbb{T} \times \mathbb{R}^2$.

- The 2-body problem for $\mu = 0$ is integrable with respect to the singular ω.

Model for these systems

(b^m-symplectic)

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

or (m-folded)

$$\omega = \mathbf{x_1^m dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Consider a system of two particles moving under the influence of the generalized potential $U(x) = -|x|^{-\alpha}$, $\alpha > 0$, with $|x|$ the distance.

Double collision

The McGehee change of coordinates used to study collisions provides b^m-symplectic and m-folded symplectic forms for any m in the problem of a particle moving in a central force field with general potential depending on m.

The 3-body problem

- Consider the system of three bodies with masses m_1, m_2, m_3 and positions $\mathbf{q_1} = (q_1, q_2, q_3), \mathbf{q_2} = (q_4, q_5, q_6), \mathbf{q_3} = (q_7, q_8, q_9) \in \mathbb{R}^3$.
- Define the 9×9 matrix $M := \text{diag}(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$.
- Assume central coordinates $(m_1\mathbf{q_1} + m_2\mathbf{q_2} + m_3\mathbf{q_3} = 0)$.
- Introduce the following "McGehee"-coordinates:

$$r := \sqrt{q^T M q}, \qquad s := \frac{q}{r}, \qquad z := p\sqrt{r}. \tag{1}$$

- $r = 0$ corresponds to triple collisions. Essentially, these are spherical coordinates since s lies on the unit-sphere in $\mathbb{R}^9$ with respect to the metric given by M.

The 3-body problem

- The standard symplectic form $\sum_{i=1}^{9} dq_i \wedge dp_i$ becomes in the new coordinates $(r, s_1, \ldots, s_8, z_1, \ldots, z_9)$.

$$\sum_{i=1}^{8}\left(\frac{s_i}{\sqrt{r}}dr \wedge dz_i + \sqrt{r}ds_i \wedge dz_i - \frac{z_i}{2\sqrt{r}}ds_i \wedge dr\right) +$$

$$+\frac{1}{\sqrt{\overline{m}_9 r\mu}}\left(\mu dr \wedge dz_9 - r\sum_{i=1}^{8}\overline{m}_i s_i ds_i \wedge dz_9 + \frac{z_9}{2}\sum_{i=1}^{8}\overline{m}_i s_i ds_i \wedge dr\right),$$

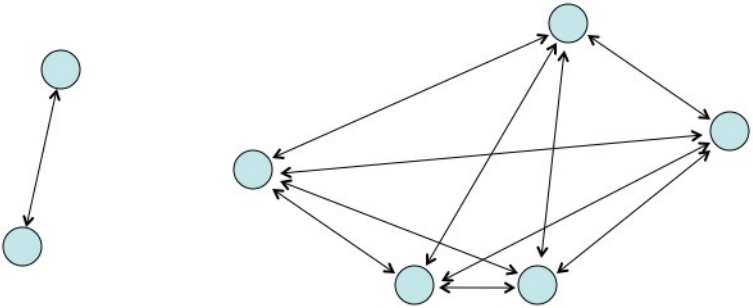
with $\mu := 1 - \sum_{i=1}^{8} s_i^2 \overline{m}_i$.

- $$\bigwedge_{i=1}^{9} dq_i \wedge dp_i = \sqrt{\frac{\mu r^7}{\overline{m}_9}} ds_1 \wedge dz_1 \wedge ds_2 \wedge dz_2 \wedge \ldots \wedge ds_8 \wedge dz_8 \wedge dr \wedge dz_9,$$

It is a $\frac{7}{2}$-folded symplectic structure. (In the n-body problem m-folded symplectic for a certain m).

Other examples

- Kustaanheimo-Stiefel regularization for n-body problem $\rightsquigarrow$ folded-type symplectic structures



- two fixed-center problem via Appell's transformation (Albouy) $\rightsquigarrow$ combination of folded-type and b^m-symplectic structures $\rightsquigarrow$ Dirac structures.

Physical examples and (singular) symplectic structures

Classical Hamiltonian systems	• Symplectic structures
Elliptic restricted 3-body problem in Mc-Gehee coordinates	• b^3-symplectic structure
McGehee regularization 3-body problem	• Folded-type symplectic structures
Kustaanheimo-Stiefel regularization for n-body problem	• Folded symplectic structure
McGehee type change for double collision	• b^m-symplectic structures • Folded-m symplectic structures

b-Symplectic/b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

b-symplectic, log-symplectic

Batakidis, Braddell, Cardona, Cavalcanti, Delshams, Frejlich, Gualtieri, Guillemin, Kiesenhofer, Klaasse, Lanius, Songhao Li, Marcut, Martínez-Torres, Miranda, Oms, Osorno, Pelayo, Pires, Planas, Radko, Ratiu, Scott, Vera, Villatoro, Weitsman

Darboux normal forms

Theorem (Guillemin-M.-Pires)

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \dots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Darboux for b^m-symplectic structures

$$\Pi = x_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

or dually

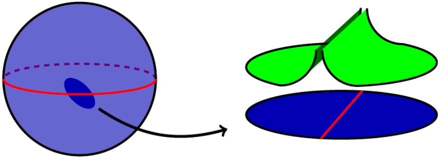
$$\omega = \frac{1}{x_1^m} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$$

Radko's classification of b-Poisson surfaces

Radko classified these structures on compact oriented surfaces:

- Geometrical invariants: The topology of S and the curves γ_i where Π vanishes.
- Dynamical invariants: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^\varepsilon(\Pi) = \int_{|h|>\varepsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \dots, \gamma_n$ and ω_Π the "dual "form to the Poisson structure.

Other classification schemes: For b^m-symplectic structures (not necessarily oriented) $\rightsquigarrow$ Scott, M.-Planas.



Other compact examples.

- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

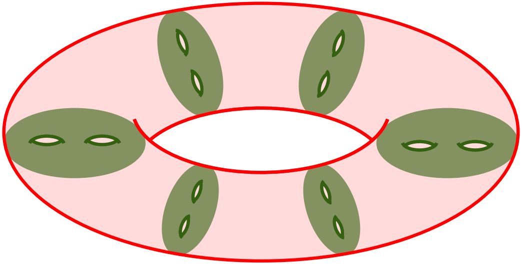
This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field transverse to the symplectic foliation induced on Z.

The singular hypersurface

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.

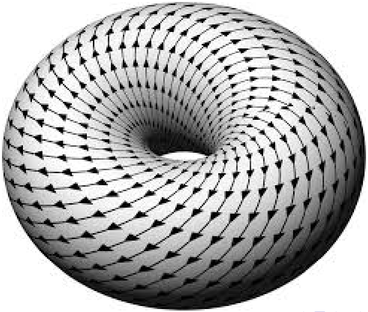


This description also works for b^m-symplectic structures.

A dual approach...

- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-cotangent bundle).
- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.
- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.

What else?

Geometrical invariants

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$^{b}H^{*}(M) \cong H^{*}(M) \oplus H^{*-1}(Z).$$

Corollary (Classification of b-symplectic surfaces à la Moser)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed,

$$^{b}H^{*}(M) \cong H^{*}_{\Pi}(M)$$

b-integrable systems

Definition

b-integrable system A set of b-functions[a] $f_1, \ldots, f_n$ on (M^{2n}, ω) such that

- $f_1, \ldots, f_n$ Poisson commute.
- $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M.

[a] $c \log |x| + g$

Example

The symplectic form $\frac{1}{h} dh \wedge d\theta$ defined on the interior of the upper hemisphere H_+ of S^2 extends to a b-symplectic form ω on the double of H_+ which is S^2. The triple $(S^2, \omega, log|h|)$ is a b-integrable system.

Example

If $(f_1, \ldots, f_n)$ is an integrable system on M, then $(\log |h|, f_1, \ldots, f_n)$ on $H_+ \times M$ extends to a b-integrable on $S^2 \times M$.

Action-angle coordinates for b-integrable systems

The compact regular level sets of a b-integrable system are (Liouville) tori.

Theorem (Kiesenhofer-M.-Scott)

Around a Liouville torus there exist coordinates $(p_1, \ldots, p_n, \theta_1, \ldots, \theta_n) : U \to B^n \times \mathbf{T}^n$ such that

$$\omega|_U = \frac{c}{p_1} dp_1 \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i, \tag{2}$$

and the level sets of the coordinates $p_1, \ldots, p_n$ correspond to the Liouville tori of the system.

Reformulation of the result

Integrable systems semilocally $\leftrightsquigarrow$ twisted cotangent lift[a] of a $\mathbb{T}^n$ action by translations on itself to $(T^*\mathbb{T}^n)$.

[a] We replace the Liouville form by $\log|p_1| d\theta_1 + \sum_{i=2}^n p_i d\theta_i$.

Proof

1. Topology of the foliation. In a neighbourhood of a compact connected fiber the b-integrable system F is diffeomorphic to the b-integrable system on $W := \mathbf{T}^n \times B^n$ given by the projections $\log p_1, p_2, \ldots, p_n$.

2. Uniformization of periods: We want to define integrals whose (b-)Hamiltonian vector fields induce a $\mathbf{T}^n$ action. Start with $\mathbf{R}^n$-action:

$$\begin{array}{rcl} \Phi : \mathbf{R}^n \times (\mathbf{T}^n \times B^n) & \to & \mathbf{T}^n \times B^n \\ ((t_1, \ldots, t_n), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(n)}_{t_n}(m). \end{array}$$

   Uniformize to get a $\mathbf{T}^n$ action with fundamental vector fields Y_i.

3. The vector fields Y_i are Poisson vector fields (check $\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = 0$).

4. The vector fields Y_i are Hamiltonian with primitives $\sigma_1, \ldots, \sigma_n \in {}^bC^\infty(W)$. In this step the properties of b-cohomology are essential. Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

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A picture...

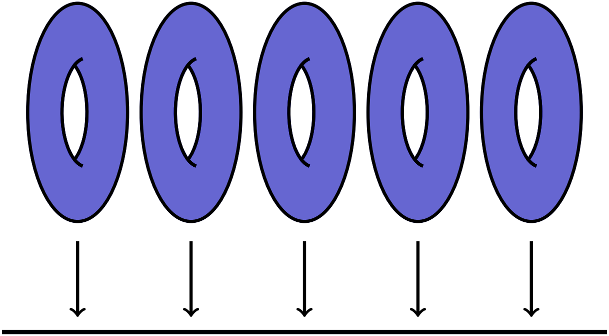


Figure: Fibration by Liouville tori

Applications to KAM theory (surviving tori under perturbations) on b-symplectic manifolds (Kiesenhofer-M.-Scott).

KAM for b-symplectic manifolds

Theorem (Kiesenhofer-M.-Scott)

Consider $\mathbf{T}^n \times B_r^n$ with the standard b-symplectic structure and the b-function $H = k \log |y_1| + h(y)$ with h analytic. If the frequency map has a Diophantine value and is non-degenerate, then a Liouville torus on Z persists under sufficiently small perturbations of H. More precisely, if $|\epsilon|$ is sufficiently small, then the perturbed system

$$H_\epsilon = H + \epsilon P$$

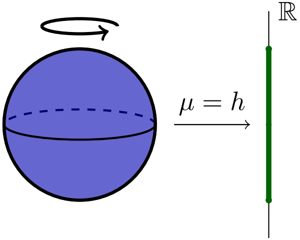
(with $P(\varphi, y) = \log |y_1| + f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1)$) admits an invariant torus $\mathcal{T}$.
Moreover, there exists a diffeomorphism $\mathbf{T}^n \to \mathcal{T}$ close to the identity taking the flow γ^t of the perturbed system on $\mathcal{T}$ to the linear flow on $\mathbf{T}^n$ with frequency vector $(\frac{k+\epsilon k'}{c}, \tilde{\omega})$.

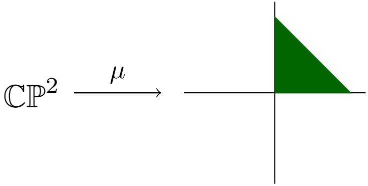
Toric manifolds and integrable systems

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

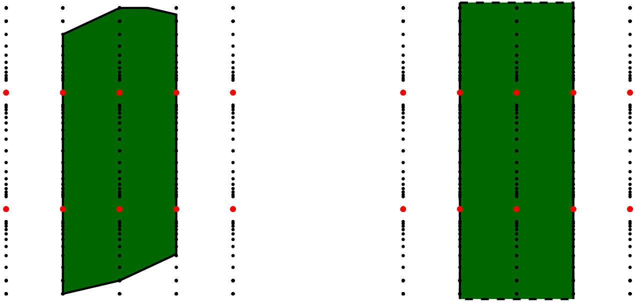
$$\begin{array}{ccc} \{\text{toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$





$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

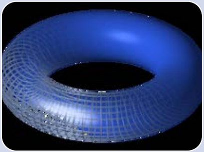
- Delzant theorem and convexity for $\mathbb{T}^k$-actions (Guillemin-M.-Pires-Scott).

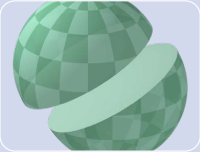


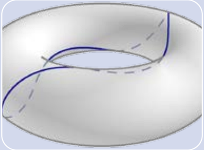
(Singular) symplectic manifolds

- b^m -Symplectic
- Symplectic
- Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

Consequences

Corollary

A manifold admitting a b^{2k}-symplectic structure also admits a symplectic structure.

Corollary

A manifold admitting a b^{2k+1}-symplectic structure also admits a folded symplectic structure.

Theorem (Cannas da Silva)

Any orientable compact 4-manifold admits a folded structure.

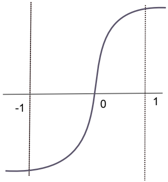
The converse is not true.

S^4 admits a folded structure but no b-symplectic structure.

Deblogging b^{2k}-symplectic structures

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta \qquad (3)$$

- Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$,



and such that outside $[-1, 1]$,

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

Deblogging b^{2k}-symplectic structures (Proof)

- Scaling:

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k-1}} f\left(\frac{x}{\epsilon}\right). \tag{4}$$

Outside the interval $[-\epsilon, \epsilon]$,

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$$

which is symplectic.

Symplectic character

- $d\alpha_i = 0 \rightsquigarrow \omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ closed.
- Outside U, ω_ϵ coincides with ω.
- In U but away from Z,

$$\omega_\epsilon^n = \frac{df_\epsilon}{dx} x^{2k} \omega^n$$

which is nowhere vanishing.

- To check that ω_ϵ is symplectic at Z, observe that

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta = \epsilon^{-2k} f'\left(\frac{x}{\epsilon}\right) dx \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta$$

which on the interval $|x| < \epsilon$ is equal to $\epsilon^{-2k}(f'(\frac{x}{\epsilon})\, dx \wedge \alpha_0 + \mathcal{O}(\epsilon)) + \beta$ and hence

$$\omega_\epsilon{}^n = \epsilon^{-2k}(f'\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0 \wedge \beta^{n-1} + \mathcal{O}(\epsilon))$$

which is non-vanishing for ϵ sufficiently small $dx \wedge \alpha_0 \wedge \beta^{n-1} \neq 0$.

Convergence

- To check

$$\omega_\epsilon^{-1} = \epsilon^{2k} g\left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \quad (5)$$

where $g(x) = \frac{1}{f'(x)}$, converges to

$$\omega^{-1} = x^{2k} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \quad (6)$$

as ϵ tends to zero.

Consider $h(x) = (\frac{d}{dx})^{2k-1} g(x)$.

- Then ω_ϵ^{-1} converges to ω^{-1} in the C^{2k-1} topology if $\epsilon h\left(\frac{x}{\epsilon}\right)$ converges in the uniform norm to $2kx$. But $x^{2k} = \epsilon^{2k} g\left(\frac{x}{\epsilon}\right)$ for $|x| > \epsilon$, so for $\epsilon < |x|$, $\epsilon h\left(\frac{x}{\epsilon}\right)$ is equal to $2kx$, and for $\epsilon > |x|$ both functions are bounded by a constant multiple of ϵ.
- Hence $\epsilon h\left(\frac{x}{\epsilon}\right)$ converges in the uniform norm to $2kx$ when $\epsilon \to 0$. and this gives the C^{2k-1}-convergence of (5) to (6).

Applications of deblogging

- Convexity of the moment map image for b^m-symplectic manifolds (GMW3).
- Quantization of b^m-symplectic manifolds (GMW4).
- Action-angle coordinates and refinement of KAM theorem (joint with Arnau Planas).
- Existence of b^{2k}-contact forms and Weinstein's conjecture. (joint with Cédric Oms).
- The quest of periodic orbits. (ongoing)
- Applications to celestial mechanics. (ongoing with Roisin Braddell, Robert Cardona, Amadeu Delshams, Jacques Féjoz, Cédric Oms, Michael Orieux)

New toy: Deblogging integrable systems

Denote $F_\epsilon^{m-i}(x) = (\frac{d}{dx} f_\epsilon(x)) x^i$, and hence $F_\epsilon^i(x) = (\frac{d}{dx} f_\epsilon(x)) x^{m-i}$.
The desingularized ω_ϵ reads

$$\omega_\epsilon = \sum_{i=0}^{m-1} F_\epsilon^{m-i}(x) dx \wedge \alpha_{m-i} + \beta.$$

Definition

The desingularization of a b^m-integrable system $\mu = (f_1, \ldots, f_n)$ is given by:

$$\mu = (f_1 = c_0 \log(x) + \sum_{i=1}^{m-1} c_i \frac{1}{x^i}, \ldots, f_n) \mapsto \mu_\epsilon = (f_{1\epsilon} = \sum_{i=1}^{m} \hat{c}_i G_\epsilon^i(x), f_2, \ldots, f_n)$$

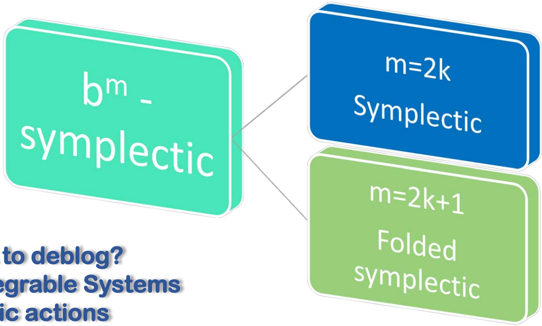
with $G_\epsilon^i(x) = \int_0^x F_\epsilon^i(\tau) d\tau$, and $\hat{c}_1 = c_0$ and $\hat{c}_{i-1} = -i c_i$ if $i \neq 0$.

Key point: The Hamiltonian vector fields are the same.

Limits

When ϵ tends to 0, μ_ϵ tends to μ.

Deblogging everything...



What to deblog?

- **Integrable Systems**
- **Toric actions**
- **Forms..**

KAM for b^{2k}-symplectic manifolds

A 3-step proof of KAM:

1. Desingularize your integrable system.
2. Action-angle coordinates go to action-angle coordinates.
3. Apply the KAM theorem for symplectic manifolds to get surviving tori.

Problem

For certain perturbations it is not possible to find a new Hamiltonian function such that $X_H^{\omega} = X_{\hat{H}}^{\omega_\epsilon}$ restrict to admissible perturbations.

Convergence of the Kolmogorov set

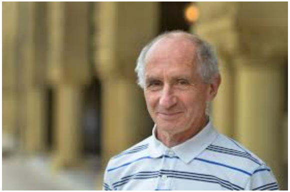
The measure of the surviving tori (Kolmogorov set) is of order $\sqrt{\delta_\epsilon}$ (parameter of the perturbation). Check the limit.

Existence of contact structures

All 3-dimensional manifolds are contact (Martinet-Lutz) in higher dimensions.

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.



Existence of b^{2k}-contact structures

Theorem (M-Oms)

For any pair (M, Z) of contact manifold and convex hypersurface there exists a b^{2k}-contact structure for all k having Z as critical set.

Corollary (of Giroux theorem)

For any 3-dimensional manifold and any generic surface Z, there exists a b^{2k}-contact structure on M realizing Z as the critical set.

What about periodic orbits?

Weinstein's conjecture 

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

Taubes proved it in dimension 3.

Theorem (M-Oms)

Given a b^{2k}-contact manifold with convex critical set Z, there exists a family of contact forms agreeing with a b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The Reeb vector fields R_ϵ converges to R^α.

Theorem (M-Oms)

Let (M, α) be a closed b^{2k}-contact manifold of dimension 3, then there exists a family of periodic orbits $\mathcal{O}_\epsilon$ associated to the Reeb vector fields R_ϵ.

A variational principle to detect periodic orbits.

Periodic orbits on $M \rightsquigarrow$ smooth maps $x : \mathbb{R}/\mathbb{Z} \longrightarrow M$ This set is called the loop space, $\mathcal{L}M$.
If $\Pi_2(M) = e$ the action functional is well-defined:

$$\mathcal{A}_H(x) := -\int_D u^*\omega + \int_0^1 H_t(x(t))\,dt,$$

(where u is an extension of x to the disk and we assume $H_t = H_{t+1}$)

Theorem

A loop x is a critical point of the action functional $\mathcal{A}_H(x)$ if and only if $t \mapsto x(t)$ is a periodic solution of the Hamiltonian system

$$\dot{x} = X_t(x(t)).$$

Key point:

$$d\mathcal{A}_H(x)(Y) = \int_0^1 \omega(\dot{x} - X_t(x(t)), Y)dt.$$

$\forall Y$ and ω is non-degenerate and this works in the b-symplectic case too.

Non-smooth periodic orbits

Find periodic "solutions" that go to infinity in the restricted three body problem.
Problem: for odd m periodic orbits are not well-understood for folded symplectic manifolds. In this case one may need to unfold (Cannas-Guillemin-Woordward) to associate a family of symplectic manifolds.