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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Rosa Vargas **Email/Phone:** rmvargas@ciencias.unam.mx

Speaker's Name: Natalia Tronko

Talk Title: Geometrical methods for reduced Hamiltonian models in plasma physics.

Date: 08 / 17 / 18 **Time:** 3 : 30 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences: In this talk, Natalia Tronko presented the derivation of the reduced Kinetic models issued from the Lagrangian and Hamiltonian formalism together with numerical results resuming implementation of exactly conserved properties. She also explained that magnetized fusion plasmas represent complex multi-scaled systems in space and time. She also talked about the advantages of using geometrical methods and in particular Hamiltonian framework for the derivation of reduced models

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- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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N.Tronko [1] in collaboration with A.Bottino [1] C. Chandre [3],E.Lanti[2]
E.Sonnendrücker [1] and L.Villard [2]

EUROfusion

Geometrical methods for reduced Hamiltonian models in plasma physics

Connection for women, MSRI, Berkeley, USA

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[2] **EPFL-SPC** Lausanne, Swiss
[3] **CNRS and Aix-Marseille University**, France

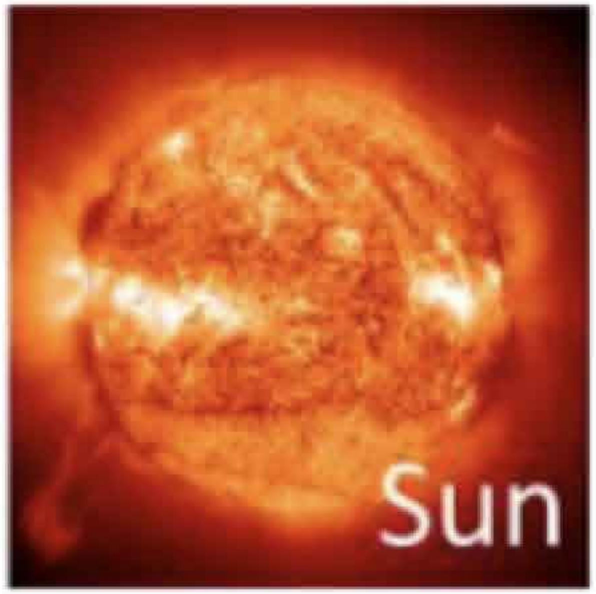
This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

Strongly magnetized plasmas

- **Plasma: 4^{th} state of the matter:** hot gas, in which thermic motion is strong enough to separate ions and electrons interacting via EM fields

- **Strongly magnetized plasma:** charged particles rotates very fast around magnetic field lines: *cyclotronic motion*

- **Magnetically confined plasmas:** the gyration radius (ρ_L) is much smaller than the size of the system (a)



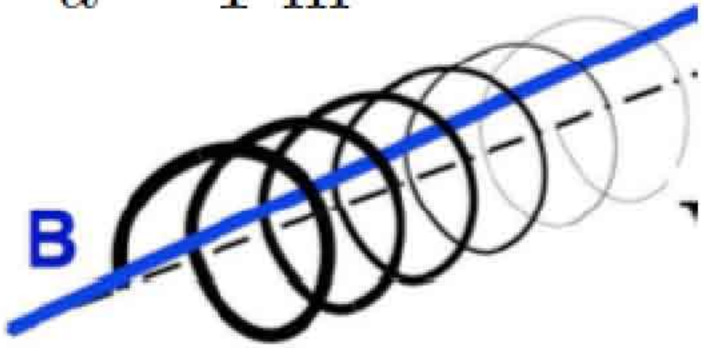
$p + p$

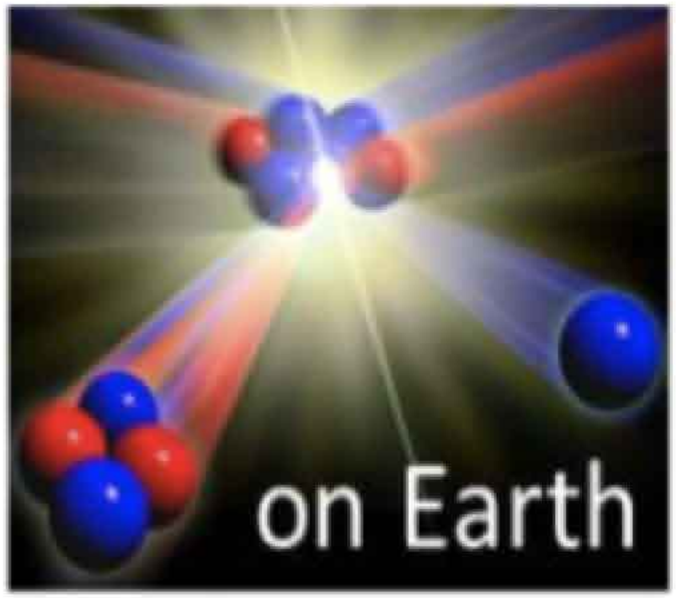
$\omega \approx 1\text{KHz}$

$\Omega_{ci} = 95.7\text{MHz}$

$\rho_{Li} \approx 1\text{cm}$

$a = 1\ \text{m}$





- **Fusion reaction:** release energy by creating from light Hydrogen isotops heavier elements

$D - T : He^4(3.5MeV) + n(14MeV$

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Laboratory devices

Challenge: bring energy from the Sun to the Laboratory

- **New source of energy**
- **Goal:** self-sustained controlled fusion reaction

- **Variety of magnetic configurations**
  - Tokamak (toroidal geometry)
  - Stellerator (twisted magnetic field lines)

- **Challenge:** Multi-scaled, Multi-species dynamics in space and time governed by **turbulence: space-time chaos**

$$\frac{m_i}{m_e} = 2 * 1.83 * 10^3$$

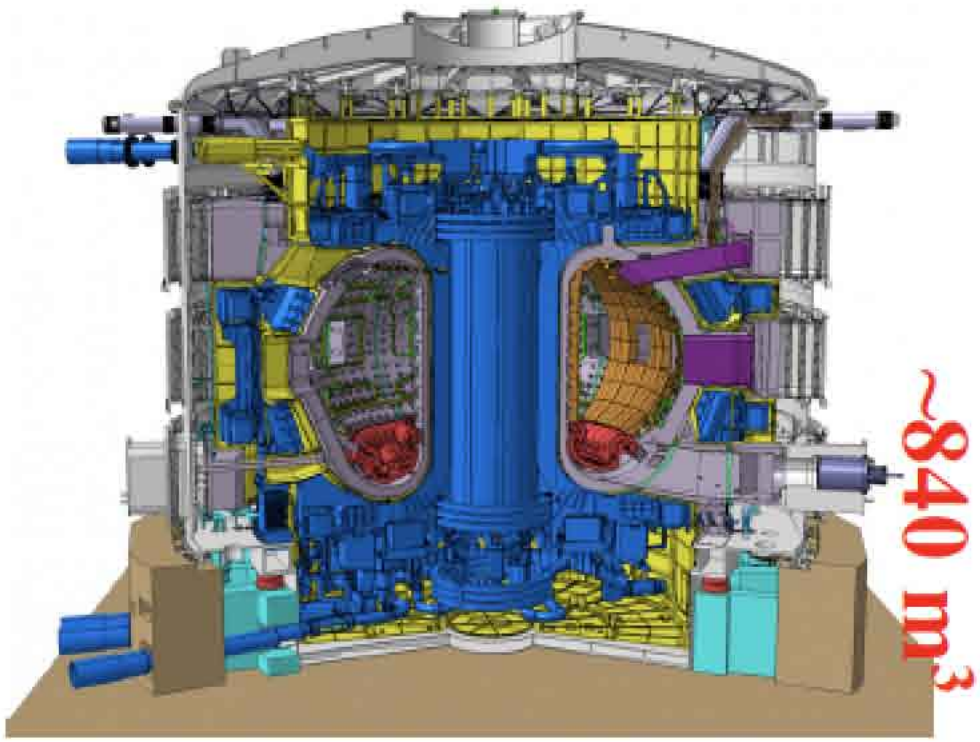
$$\frac{\rho_{Li}}{a} \approx 10^{-3}$$

$$\frac{\omega}{\Omega_{ci}} \approx 10^{-3}$$

$$\frac{\omega}{\Omega_{ce}} \approx 10^{-6}$$

Wendelstein 7 X, Greifswald, Germany

Plasma volume ~30 m^3



ITER, Cadarache, France

Plasma volume ~840 m^3

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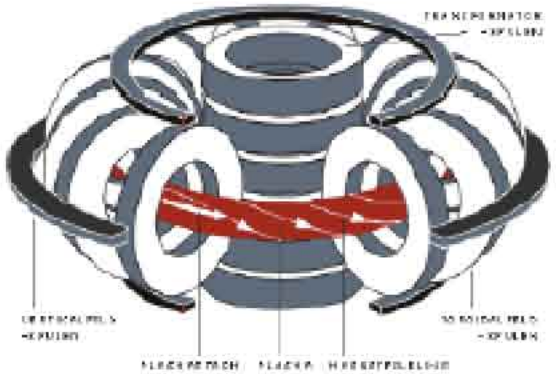
Fusion plasma technical challenge

- Magnetic field 10.000 stronger than on the Earth
- Plasma temperature 100 Millions degrees Celsius
- Plasma density 250 000 times thinner then the Earth's mantel
- Requires ultra-robust costly materials

Ignition criterion: no external heating needed to maintain fusion reaction

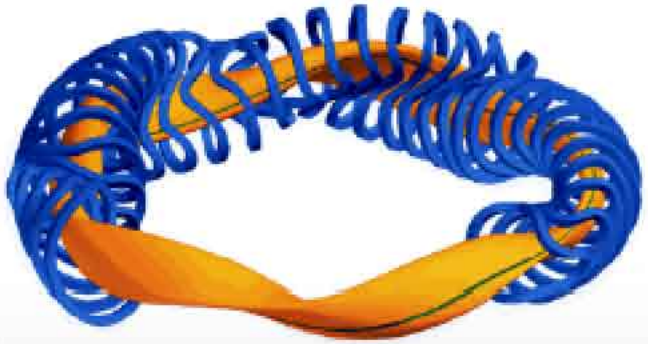
Temperature	10^8 degrees Celsius
Density	10^{20} m^{-3}
Energy Confinement time	2 sec

$$n\tau_E T \geq 2 \times 10^{28} m^{-3} s \ {}^\circ\mathrm{C}$$



Toroidal magnetic configuration:
Tokamak JET in Culham UK

$$n\tau_E T \approx 0.4 \times 10^{28} m^{-3} s {}^\circ\mathrm{C}$$



Stellerator configuration:
Wendelstein 7X Greifswald,
Germany: **new record June 2018, Nature**

$$n\tau_E T \approx 0.03 \times 10^{28} m^{-3} s {}^\circ\mathrm{C}$$

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Sources of deconfinement

Free sources of energy

∇T

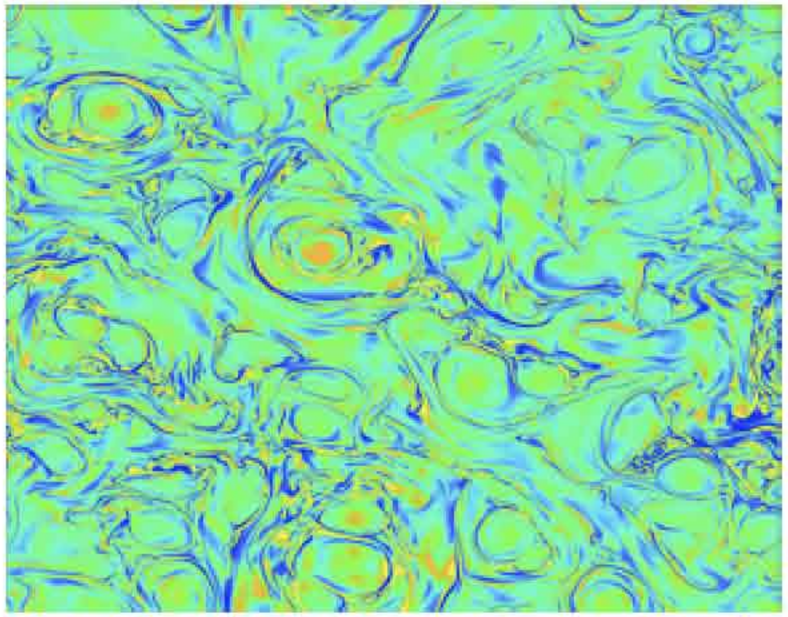
∇n

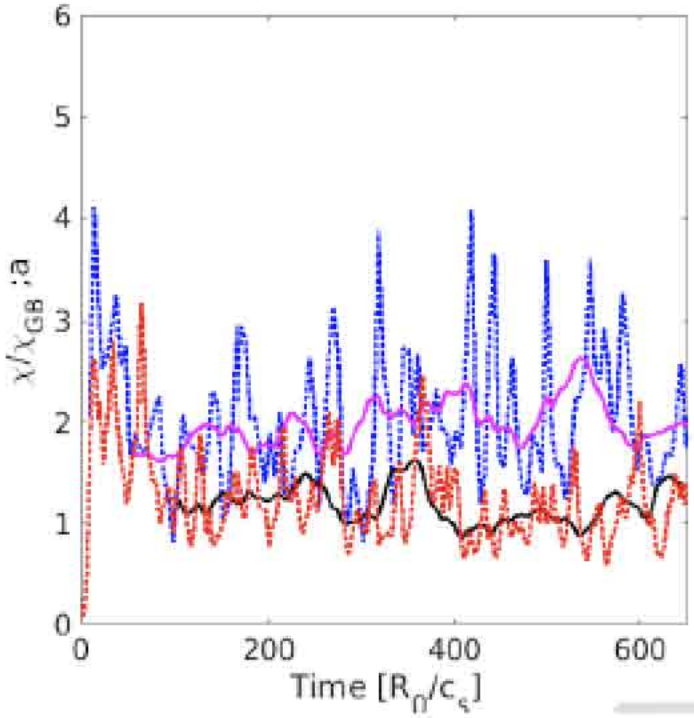
Microinstabilities (exponential growth of EM field fluctuations)

$\delta B \sim \delta B_0 \, e^{\gamma t} \qquad \delta E \sim \delta E_0 \, e^{\gamma t}$

Plasma turbulence: Low-frequency

$$\omega \ll \Omega_{ci} = \frac{q_i B}{m_i} = 9.571 \times 10^7 \ s^{-1}$$





Turbulent (anomaleous) transport

Plasma deconfinement

Improvement of experimental set up requires numerical modeling

Computational challenges

- Direct approach:

- Simulating 10^{23} particles interacting by mean of electromagnetic field

Technical requirements: 500 Milliards of TB of data storage=5* 10^{21} Bytes= 5 Million PetaByte:

  - 10 days of calculation on SUMMIT Top 500 of Supercomputers in the world (Oak Ridge National Lab)

Modeling Plasma Turbulence: realistic scenario

A model

- containing essential physical mechanisms driving turbulence
- **robust mathematical structure and conservation properties**

↓

Hamiltonian and Lagrangian description in order to control quality of numerical simulations **are essential**

Vlasov-Maxwell Hamiltonian system

Replace a particle (**x**,**v**) by a probability density on the phase space f(**x**,**v**):

Kinetic description: essential for resonant field/particles interactions

Morrison 1980

Marsden Weinstein 1982

- **Phase space**

$f(\mathbf{x},\mathbf{v},t)$ with constraints $\nabla\cdot\mathbf{B}=0$

$\mathbf{E}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t)$

- Poisson equation $\nabla\cdot\mathbf{E}=4\pi\sum_{\rm sp}\int \mathrm{d}^3\mathbf{v}\,f(\mathbf{x},\mathbf{v},t)$

- **Hamiltonian**

$$H[\mathbf{E},\mathbf{B},f]=\frac{1}{2}\sum_{\rm sp}\int \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{v}\ f(\mathbf{x},\mathbf{v},t)\ m_{\rm sp}v_{\rm sp}^2+\frac{1}{8\pi}\int \mathrm{d}^3\mathbf{x}\left(\mathbf{E}^2+\mathbf{B}^2\right)$$

- **Non-canonical Poisson bracket**

$$[\mathsf{F},\mathsf{G}]=$$

1) Particle bracket

$$\int \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{v}\ f\left(\frac{\partial}{\partial\mathbf{x}}\frac{\delta\mathsf{F}}{\delta f}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{\delta\mathsf{G}}{\delta f}-\frac{\partial}{\partial\mathbf{x}}\frac{\delta\mathsf{G}}{\delta f}\cdot\frac{\partial}{\partial\mathbf{v}}\frac{\delta\mathsf{F}}{\delta f}\right)$$

2) Field bracket

$$+\int \mathrm{d}^3\mathbf{x}\ \left(\frac{\delta\mathsf{F}}{\delta\mathbf{E}}\cdot\nabla\times\frac{\delta\mathsf{G}}{\delta\mathbf{B}}-\frac{\delta\mathsf{G}}{\delta\mathbf{E}}\cdot\nabla\times\frac{\delta\mathsf{F}}{\delta\mathbf{B}}\right)$$

3) Coupling bracket

$$\int \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{v}\ \left(\frac{\delta\mathsf{F}}{\delta\mathbf{E}}\cdot\frac{\partial f}{\partial\mathbf{v}}\frac{\delta\mathsf{G}}{\delta f}-\frac{\delta\mathsf{G}}{\delta\mathbf{E}}\cdot\frac{\partial f}{\partial\mathbf{v}}\frac{\delta\mathsf{F}}{\delta f}\right)+\int \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{v}\ f\ \mathbf{B}\cdot\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta\mathsf{F}}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta\mathsf{G}}{\delta f}\right)$$

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Vlasov-Maxwell Hamiltonian system

- **Equations of motion (for one of the species)**

$$\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} = [H, \mathbf{E}] = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}$$

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = [H, \mathbf{B}] = -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} - \frac{4\pi e}{c}\int \mathrm{d}^3\mathbf{v}\ \mathbf{v}\ f(\mathbf{x}, \mathbf{v}, t)$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [H, f] = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{v}} + \left(\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{x}}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \frac{4\pi e}{c}\int \mathrm{d}^3\mathbf{v}\ \mathbf{v}\ f(\mathbf{x}, \mathbf{v}, t)$$

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{v}} - \left(\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{x}}$$

- **Multi-species challenge**

Electrons are 1.83×10^3 lighter than ions! →

Adiabatic limit

$$m_i \to \infty$$

Electrons modelized by fluid

Simulations for both species (D and e) are required to achieve realistic scenario
Time step needs to be decreased as comparing to the case of adiabatic (cold) electrons

$$dt_{\text{kin}} \sim \sqrt{\frac{m_e}{m_i}} \sim \frac{1}{60}\ dt_{\text{adiab}}$$

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Eulerian and Lagrangian approaches for kinetic simulations

Lagrangian code Particle-In-Cell

Eulerian code, grid-based

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}$$

$$\boxed{\frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0}$$

- Reconstruct Vlasov dynamics from the particle characteristics
- Fields: treated on the grid: final elements

- Macro-Particles in the phase-space

- **Noise issue: need 10^6 markers at least!**

- Grid approach: direct discretisation of the distribution function f together with fields

- **CFL limit of the time step and space resolutions** : limiting numerical configurations

$$C = \frac{u\Delta t}{\Delta x} \leq 1$$

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Difficulties of kinetic simulations

- **The Vlasov-Maxwell model is well known but still be unsuitable for realistic numerical simulations**

- Storage problem for 6D distribution function:
  - 1 point in time 2,5 GB in **6D (x,v)**: (150x64x16)x(16x64x16)
  - Realistic simulation with kinetic electrons: $\omega_{ei}=1.75\text{x}10^{11}$ sec^{-1}
  - TCV energy confinement time $\tau_E= 2\text{x}10^{-2}$ sec will require Ntimes_steps= 3,5 x10^9
  - 800*10^6 TB of storage
  - Space available on Supercomputer Marconi: 1TB pro Project!

- Computational resources**: time resolution is limited by cyclotron frequency space resolution is limited by Debye length 10^{-4} m!**

- **Reduction of kinetic model :**
- Adapting dynamic coordinates with respect to physical properties of turbulence
- Store only energy and other moments of the distribution function

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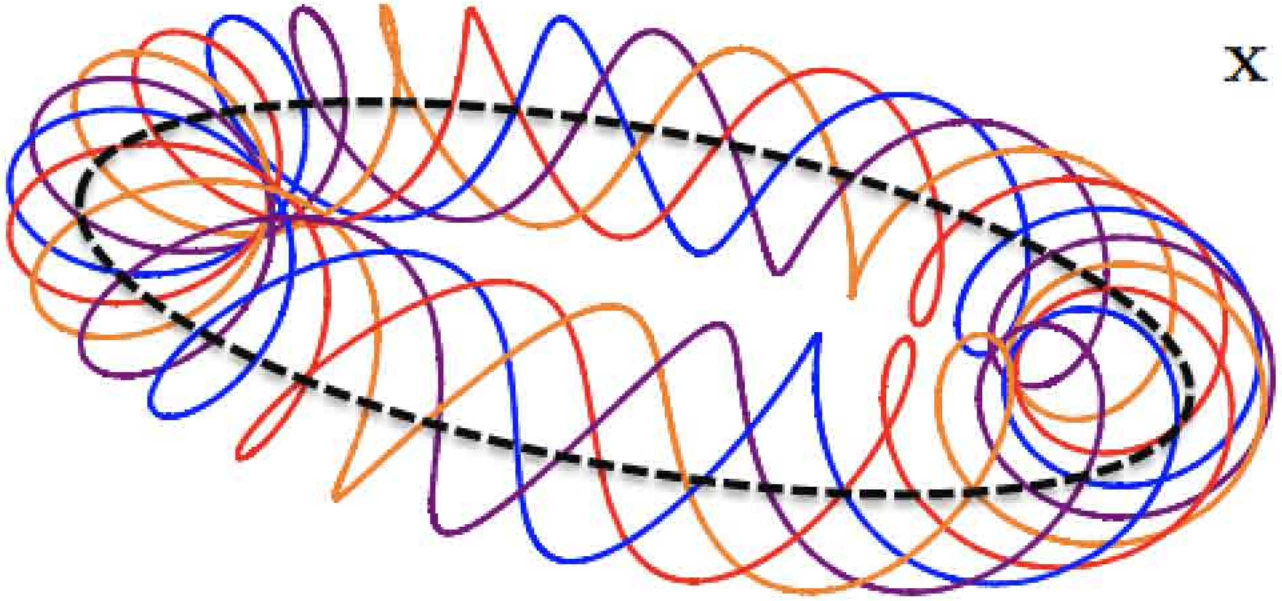
What is Gyrokinetic theory?

Idea: Use physics as a guidance for low frequency Maxwell-Vlasov dynamical reduction

$$\epsilon_\omega = \frac{\omega}{\Omega_{ci}} \sim 10^{-3}$$

1. Replacing particle with position x by the guiding-center: instantaneous center of rotation X around magnetic field lines



$$\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}_0$$

6D $\longrightarrow$ 5D (4D+1)

$$f(\mathbf{x}, \mathbf{v}) \rightarrow f(\mathbf{X}, v_\parallel, \mu)$$

2. Scales of motion separation: use existence of fast and slow variables

Systematically eliminate fastest scale of motion irrelevant for turbulent transport: increasing dt by 1000!

- Magnetic Moment: ***adiabatic invariant*** $\mu = \frac{mv_\perp^2}{2B}$
- Gyroangle ***: fast angle*** θ

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Gyrokinetic dynamical reduction

A systematic dynamical reduction procedure such that at each step

$\dot{\mu} = 0$ Has a trivial dynamics

θ Is uncoupled

$$\left.\begin{array}{ll} \dot{\mu} = 0 & \text{Has a trivial dynamics} \\ \theta & \text{Is uncoupled} \end{array}\right\} \quad 6\text{D} \longrightarrow 4\text{D}+1, \quad f(\mathbf{x}, \mathbf{v}) \to f(\mathbf{X}, v_{\|}, \mu)$$

Simple gyroaveraging leads to loss of important information: resonant interaction between fields and particles

Goal: Invertible near identity change of coordinates

Range of small parameters raising from several aspects: geometry, physics of turbulent motion: **multi-scaled asymptotic theory**

Goal: two step

- Systematic asymptotic procedure for dynamical reduction on the particle phase space → **Hamiltonian approach**
- Systematic coupling of the reduced particle dynamics with fields → **Lagrangian approach**

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Costs of Gyrokinetic simulations

The GK codes require HPC platforms to get results in a reasonable amount of time

1 node-hour ≈ 0.4 CHF ≈ 0.4 USD

- EUROfusion projects: Marconi #18 in the world

- *HPC Budgets*
- **2018 “GKICK”** 850 000 node hours

- **2015-2017 “VeriGyro”** 1 280 000 node hours

Type of simulation	Node-hours pro run	Restarts (every 24 hours)	Time step in $1/\Omega_{ci}$	Required Storage
Adiabatic electrons	200	0	dt =50	1 GB
Linear with kinetic electrons	780	1	dt= 1	5 GB
Nonlinear with kinetic electrons	14400	2	dt =0.25	300 GB

Investing in data storage and backups is important!

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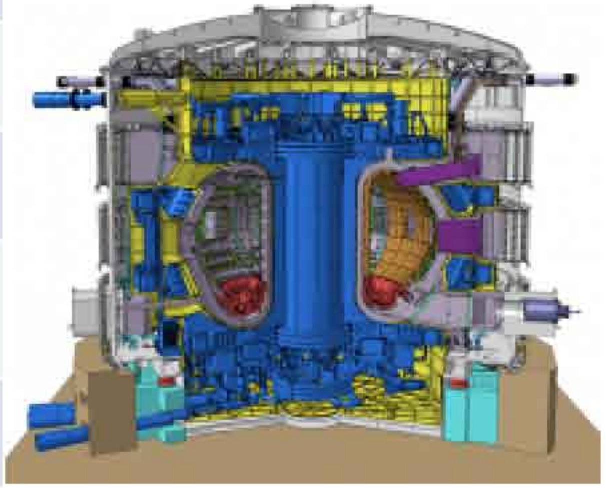
Costs of experiments

- 1 shot of TCV Tokamak in SCP Lausanne costs 1000 CHF
- 1 shot of ITER is estmated 1 000 000 CHF

	TCV	ITER
Major radius	1.54 m	6.2 m
Minor Radius	0.56 m	2.0 m
B_{Tor}	1.54 T	5 T
n	$20*10^{20}$ m^{-3}	$10*10^{20}$ m^{-3}
T_i	≤ 1KeV	8.0 KeV
T_e		
Discharge time	2.6s	400s
Plasma Heating	1 MW	40 MW
Energy gain	no	yes



Developing trustable and robust mathematical modeling is essential for success of magnetic fusion

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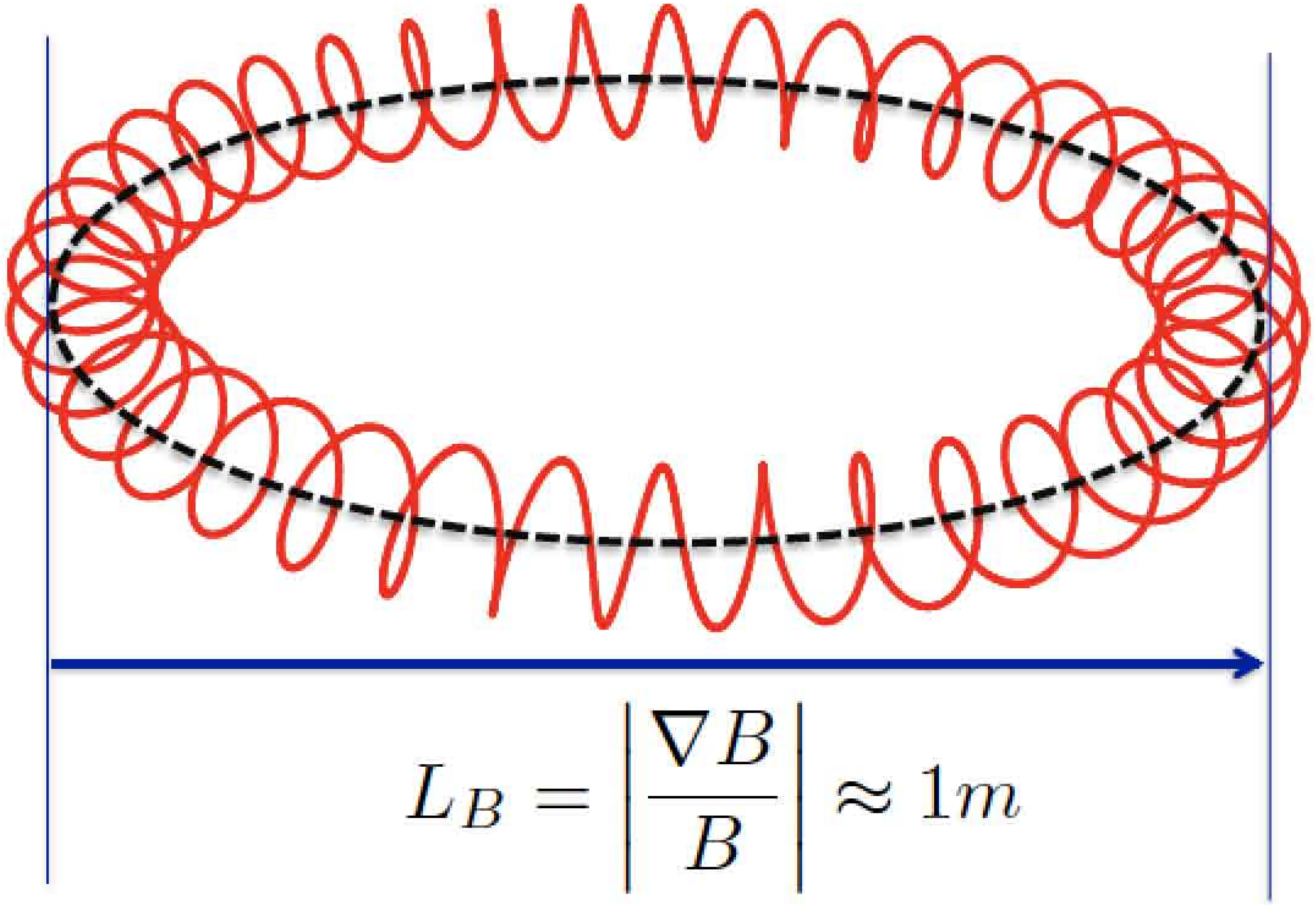
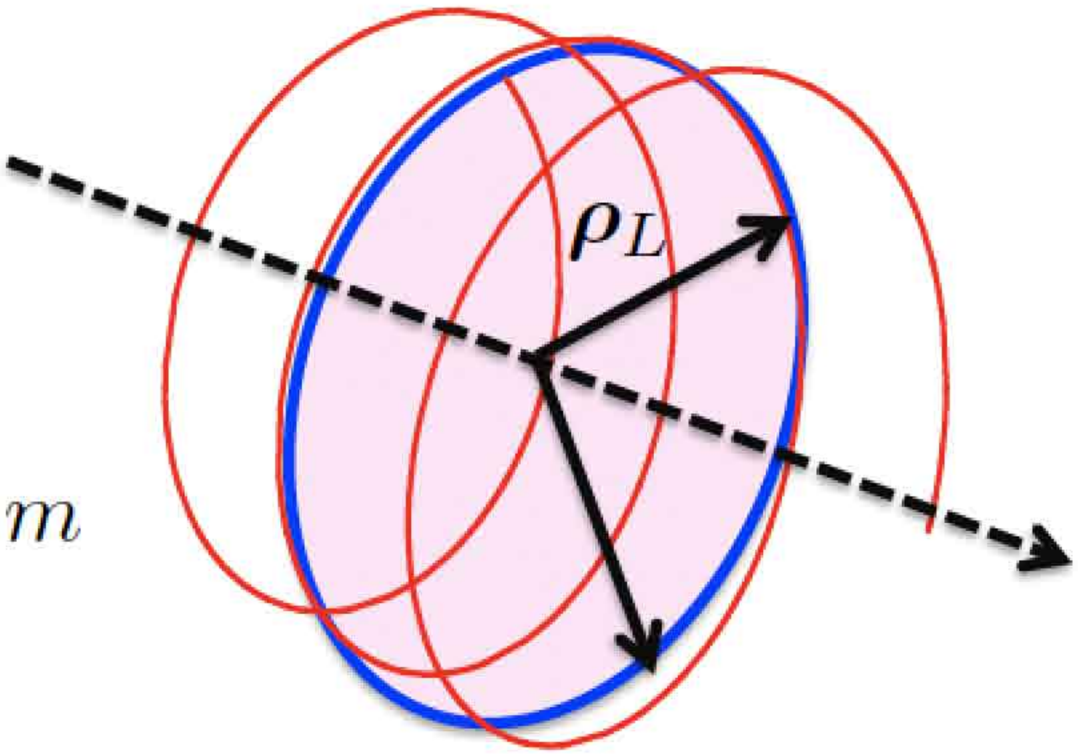
Small parameters: 1) Magnetic curvature



$$L_B = \left|\frac{\nabla B}{B}\right| \approx 1m$$

Separation of scales of motion



$$\rho_L \approx 10^{-3} m$$

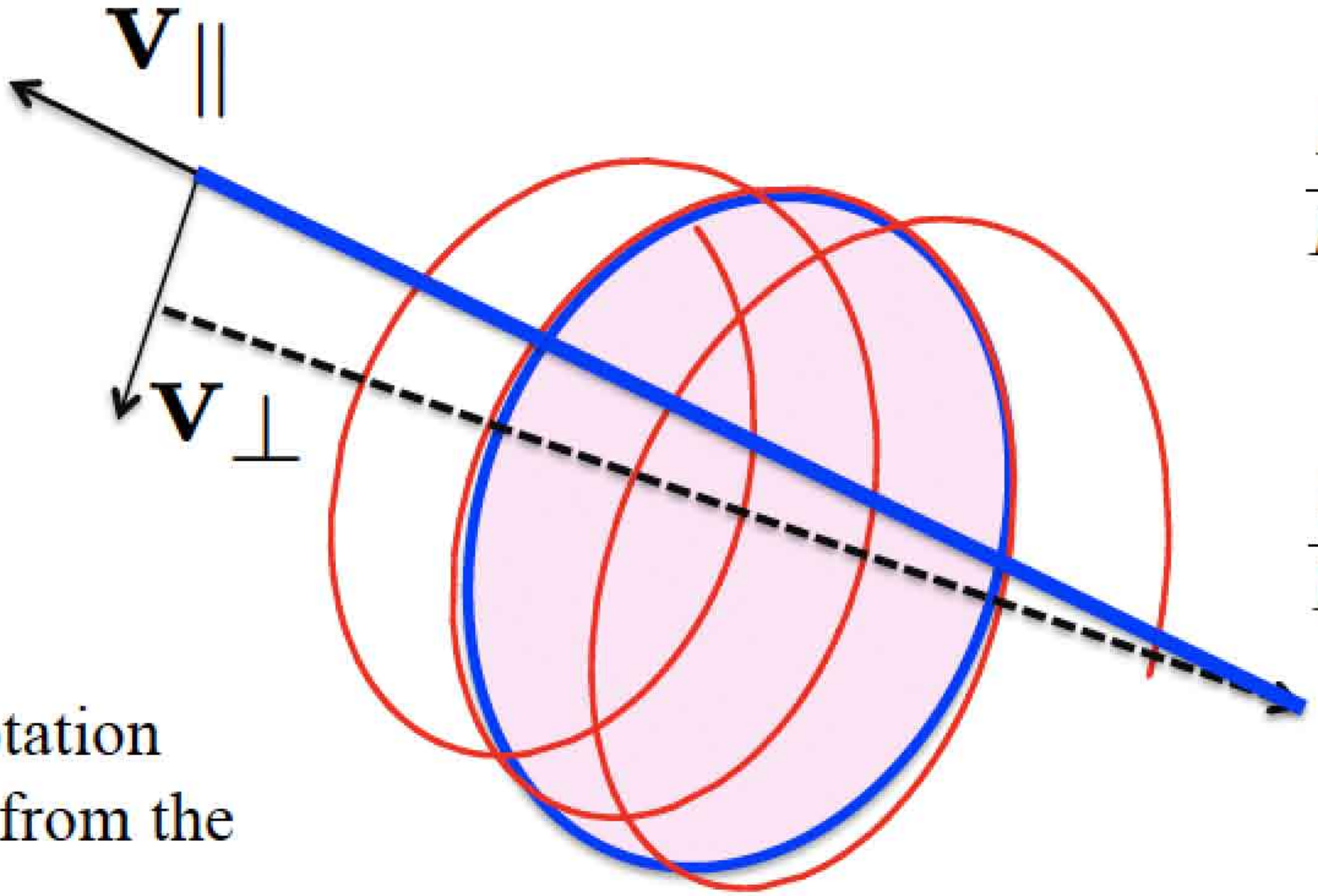
Small parameter

$$\epsilon_B = \rho_L \left|\frac{\boldsymbol{\nabla} B}{B}\right|$$

Small parameters : 2) Anisotropy of turbulence

Plasma turbulence : perpendicular to magnetic field lines

$$\epsilon_{\|} = \frac{k_{\|}}{k_{\perp}} \ll 1$$



$$\frac{E_{1\|}}{E_{1\perp}} \ll 1$$

$$\frac{B_{1\|}}{B_{1\perp}} \ll 1$$

The center of instantaneous rotation (guiding-center) slowly drifts from the magnetic field: due to

- Magnetic field curvature
- Fluctuations of electromagnetic fields (when considered)

- Parallel ion velocity $\frac{v_{i\|}}{v_{i\perp}} \approx 100$
- Parallel electron velocity $\frac{v_{e\|}}{v_{e\perp}} \approx 10^4$

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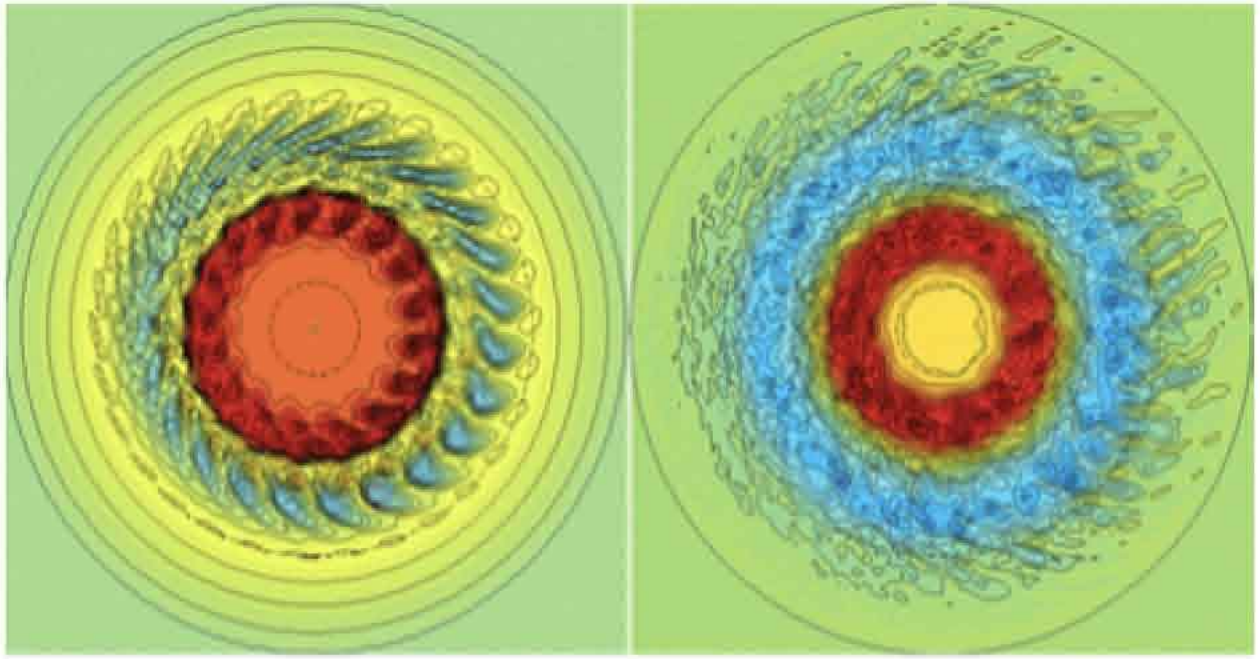
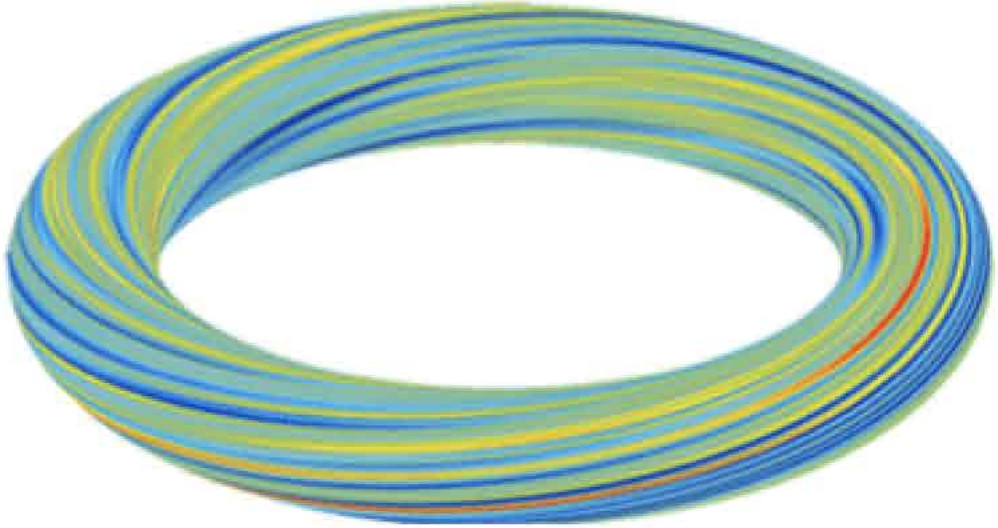
Small parameters : 2) Anisotropy of turbulence





Lagrangian simulations with ORB5 code by L. Villard, SCP Lausanne

- Fluctuations of electrostatic potential: early (left) and late (right) stage of turbulence.
- Development of short wavelength perturbations with respect to the size of the tokamak
- 3D vue: elongation of perturbations along the magnetic field lines

$k_\perp \rho_i \sim 1$ Turbulent structures are of the Larmor radius size: small scales need to be solved

$\epsilon_\delta = k_\perp \rho_i \frac{e\delta\phi}{T_i}$ Typical parameter to characterize turbulent fluctuations

GK Orderings

- **Guiding- center:** background quantities: $\epsilon_B = \rho_0 |\nabla B/B|$
- **Gyrocenter:** fluctuating fields:
- Anisotropy of turbulence

$$\left.\begin{aligned} \epsilon_\delta &= (k_\perp \rho_i)\frac{e\delta\phi}{T_i} \\ \epsilon_\omega &= \frac{\omega}{\Omega_{\mathrm{ci}}} \\ \epsilon_{||} &= k_{||}/k_\perp \ll 1 \end{aligned}\right\} \epsilon_{||} \sim \epsilon_\omega \sim \epsilon_\delta$$

- **Ordering defines physics: There is NO unique gyrokinetic model**

- Gyrokinetics

  $k_\perp \rho_i \sim 1$

- Drift-kinetics

  $k_\perp \rho_i \ll 1$

- Maximal ordering

  $\epsilon_B \sim \epsilon_\delta$

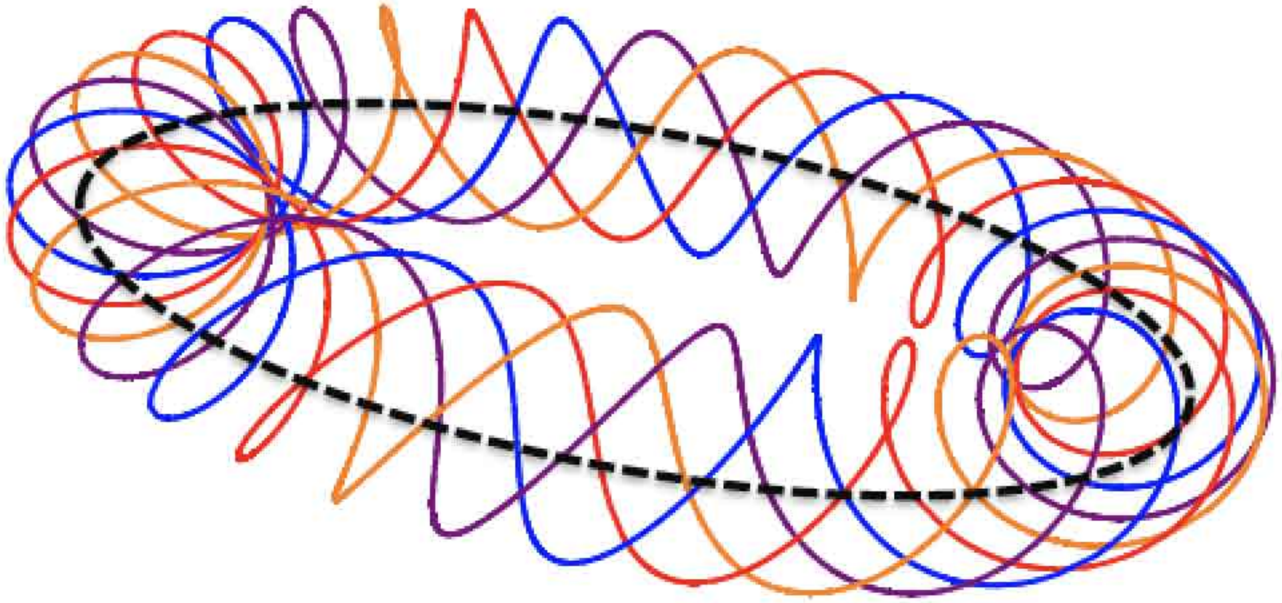
- Code ordering

  $\epsilon_B \ll \epsilon_\delta$

*[**Tronko, Chandre,** J.Plasma Phys., 2018]* $\epsilon_B = \epsilon_\delta^2$

*[**Brizard, Hahm** Rev.Mod.Phys., 2007]* $\epsilon_B = \epsilon_\delta^{3/2}$

Phase space Lagrangian formalism



- **Charged particle in external magnetic field** $\mathbf{B} = \nabla \times \mathbf{A}$

$\gamma = L_p \, \mathrm{d}t$

- Phase-space one form

$$L_p = \left(m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \right) \cdot \dot{\mathbf{x}} - H(\mathbf{x}, \mathbf{v})$$

$$H = \frac{1}{2m} |\mathbf{v}|^2$$

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_p}{\partial \dot{\mathbf{v}}} &= \frac{\partial L_p}{\partial \mathbf{v}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_p}{\partial \dot{\mathbf{x}}} &= \frac{\partial L_p}{\partial \mathbf{x}} \end{aligned}\right\} \Rightarrow$$

$$m\dot{\mathbf{v}} = \frac{e}{c}\mathbf{v} \times \mathbf{B}$$

- Lorenz force

$$\mathbf{v} = \dot{\mathbf{x}}$$

- Lagrangian constraint

- **Starting point of guiding-center reduction**

Guiding-center dynamical reduction

Splitting difficulties: first solving problem for particle motion in external non-uniform magnetic field

Goal: removing gyroangle dynamical dependencies up to the first order in ϵ_B

$$\epsilon_B = \rho_0 \left| \frac{\nabla B}{B} \right| \sim \rho_i \rho_j \left| \partial^2 \mathbf{A} / \partial \bar{x}_i \partial \bar{x}_j \right|$$

- **Physical ordering: with respect to curvature of magnetic field**

- **Exact solution in SLAB geometry exists, i.e. for** $\epsilon_B = 0$

- **Step1: infinitesimal shift in velocity space:**
  - **Second order in** ϵ_B **shift in velocity:**

$$\mathbf{w} = \mathbf{v} + \frac{e}{mc} \underbrace{\left[\mathbf{A}\left(\bar{\mathbf{x}} + \boldsymbol{\rho}_0\right) - \mathbf{A}\left(\bar{\mathbf{x}}\right) - \left(\boldsymbol{\rho}_0 \cdot \nabla\right) \mathbf{A}(\bar{\mathbf{x}}) - \frac{1}{2}\left(\boldsymbol{\rho}_0 \boldsymbol{\rho}_0 : \nabla\nabla\right) \mathbf{A}(\bar{\mathbf{x}}) \right]}_{\sim \mathcal{O}(\epsilon_B^2)}$$

$$\mathbf{w} = w_{\|} \hat{\mathbf{b}}(\mathbf{x}) + w_{\perp} \hat{\perp}(\theta, \mathbf{x})$$

  - **Particle position decomposition : instantaneous rotation center and Larmor radius** $\mathbf{x} = \bar{\mathbf{x}} + \boldsymbol{\rho}_0$

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Guiding-center gauge transformation

- **Step2:** $$\gamma = \gamma_0 + \gamma_1 + \mathrm{d}\sigma_1 + \sum_{i=2}^{4} \mathrm{d}\sigma_i + \mathcal{O}(\epsilon_B^2)$$

[Littlejohn 1983]

- **Symplectic part free from oscillations at lower oder**
- **Hamiltonian**

$$\gamma_0 = \left(\frac{e}{c}\mathbf{A}(\mathbf{x}) + m w_\parallel \widehat{\mathbf{b}}(\bar{\mathbf{x}})\right) \cdot \mathrm{d}\bar{\mathbf{x}} - H \mathrm{d}t$$

$$H = m w_\parallel^2/2 + m w_\perp^2/2$$

- **Oscillating part: aiming to eliminate up to the second order in** ϵ_B

$$\gamma_1 = \left(\boxed{\frac{e}{c}\left(\boldsymbol{\rho}_0 \cdot \boldsymbol{\nabla}\right)\mathbf{A}} + \frac{e}{2c}\left(\boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\right)\mathbf{A} + \boxed{m w_\perp \widehat{\perp}\left(\theta, \mathbf{x} + \boldsymbol{\rho}_0\right)} + m w_\parallel \left[\widehat{\mathbf{b}}(\bar{\mathbf{x}} + \boldsymbol{\rho}_0) - \widehat{\mathbf{b}}(\bar{\mathbf{x}})\right]\right) \cdot \mathrm{d}\bar{\mathbf{x}}$$

$$+ \left(\frac{e}{c}\mathbf{A} + \frac{e}{c}\left(\boldsymbol{\rho}_0 \cdot \boldsymbol{\nabla}\right)\mathbf{A} + \frac{e}{2c}\left(\boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\right)\mathbf{A} + m w_\perp \widehat{\perp}\left(\theta, \mathbf{x} + \boldsymbol{\rho}_0\right) + m w_\parallel \widehat{\mathbf{b}}(\bar{\mathbf{x}} + \boldsymbol{\rho}_0)\right) \cdot \mathrm{d}\boldsymbol{\rho}_0$$

- **Defining Larmor radius, eliminating** $\mathcal{O}(\epsilon_B^0)$ **oscillating terms**

$$\boxed{\boldsymbol{\rho}_0 = \frac{m w_\perp c}{e B}\widehat{\boldsymbol{\rho}}}$$

- **First gauge transformation** $$\sigma_1 = -\frac{e}{c}\mathbf{A} \cdot \boldsymbol{\rho}_0 - \frac{e}{2c}\left(\boldsymbol{\rho}_0 \cdot \boldsymbol{\nabla}\right)\mathbf{A} \cdot \boldsymbol{\rho}_0 - \frac{e}{6c}\left(\boldsymbol{\rho}_0\boldsymbol{\rho}_0 : \boldsymbol{\nabla}\boldsymbol{\nabla}\right)\mathbf{A} \cdot \boldsymbol{\rho}_0$$

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Guiding-center gauge transformation

- **Step 3: Near identity coordinate transformation**

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{X} + \boldsymbol{\xi}(\mathbf{X}, W_{\|}, W_{\perp}, \Theta) \\ w_{\|} &= W_{\|} + \mathcal{W}_{\|}(\mathbf{X}, W_{\|}, W_{\perp}, \Theta) \\ w_{\perp} &= W_{\perp} + \mathcal{W}_{\perp}(\mathbf{X}, W_{\|}, W_{\perp}, \Theta) \\ \theta &= \Theta + \mathcal{T}(\mathbf{X}, W_{\|}, W_{\perp}, \Theta) \end{aligned}$$

$$\boldsymbol{\xi} \sim \mathcal{W}_{\|} \sim \mathcal{W}_{\perp} \sim \mathcal{T} \sim \mathcal{O}(\epsilon_B)$$

- **Gauge transformation defines Θ-independent symplectic form and new coordinates up to the second order in** ϵ_B

$$\Gamma = \gamma_0^{\mathrm{sympl}} + \gamma_1 + \sum_{i=1}^{4} \mathrm{d}\sigma_i = \left(\frac{e}{c}\mathbf{A}(\mathbf{X}) + mW_{\|}\widehat{\mathbf{b}}(\mathbf{X}) - \frac{mc}{e}\mu\mathbf{R}\right) \cdot \mathrm{d}\mathbf{X} + \frac{mc}{e}\mu\mathrm{d}\Theta$$

$$\mathbf{R} = \nabla\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2 \quad \sim \mathcal{O}(\epsilon_B)$$

$$\mu = \frac{mW_{\perp}^2}{2B^2}$$

- **Hamiltonian still being Θ-dependent, therefore final coordinate change is required**

$$H = \frac{1}{2}mW_{\|}^2 + \frac{1}{2}mW_{\perp}^2 + m(W_{\|}\mathcal{W}_{\|} + W_{\perp}\mathcal{W}_{\perp}) + \mathcal{O}(\epsilon_B^2) \qquad \Theta\text{-dependent}$$

Last step: canonical Lie-transform on Hamiltonian to remove Θ-dependency

Guiding-center Poisson bracket

- **Inverting the symplectic matrix**

$$\omega = \mathrm{d}\Gamma = \omega_{ij}\mathrm{d}z^i \wedge \mathrm{d}z^j$$

→

$$\{F, G\}_{\mathrm{gc}} = \frac{\partial F}{\partial z^i}(\omega^{-1})^{ij}\frac{\partial G}{\partial z^j}$$

$$\{F, G\}_{\mathrm{gc}} = \frac{e}{mc}\left(\frac{\partial F}{\partial \theta}\frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu}\frac{\partial G}{\partial \theta}\right) + \frac{\mathbf{B}^*}{mB^*_\parallel}\cdot\left(\boldsymbol{\nabla}F\frac{\partial G}{\partial W_\parallel} - \frac{\partial F}{\partial W_\parallel}\boldsymbol{\nabla}G\right) - \frac{c\widehat{\mathbf{b}}}{eB^*_\parallel}\cdot(\boldsymbol{\nabla}F \times \boldsymbol{\nabla}G)$$

- **Adiabatic invariant contains the whole coordinate transformation**

$$\mu = \frac{mW_\perp^2}{2B^2}$$

- **Symplectic magnetic field**

$$\mathbf{B}^* = \mathbf{B} + \frac{mc}{e}W_\parallel\boldsymbol{\nabla}\times\widehat{\mathbf{b}} + \mathcal{O}(\epsilon_B^2)$$

Canonical Guiding-center Lie transform

- **Formal scales separation in the Poisson bracket**

$$\{F,G\}_{\rm gc} = \{F,G\}_{-1} + \{F,G\}_{0} + \{F,G\}_{1}$$

- **Canonical Lie-transform (infinitesimal transformation)**

$$\bar{H}(\mathbf{z}) = \mathrm{e}^{-\pounds_S} H(\mathbf{Z}) = H - \{S,H\}_{\rm gc} + \frac{1}{2}\{S,\{S,H\}_{\rm gc}\}_{\rm gc} + \mathcal{O}(S^3)$$

- **Coordinate transform is constructed simultaneously with reduced Hamiltonian**

$$\mathbf{z} = \mathrm{e}^{\pounds_S}\mathbf{Z} \quad \Longrightarrow \quad \begin{aligned} \bar{\mathbf{x}} &= \mathbf{X} + \boldsymbol{\xi} + \{S,\mathbf{X}\}_{\rm gc} \\ w_{\|} &= W_{\|} + \mathcal{W}_{\|} + \{S,W_{\|}\}_{\rm gc} \\ w_{\perp} &= W_{\perp} + \mathcal{W}_{\perp} + \{S,W_{\perp}\}_{\rm gc} \\ \theta &= \Theta + \mathcal{T} + \{S,\Theta\}_{\rm gc}, \end{aligned}$$

- **Scalar invariance** $\bar{H}(\mathbf{z}) = H(\mathbf{Z})$

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Guiding-center reduced dynamics

- **Guiding-center generating function**

$$S_{\rm gc} = \quad \frac{m^3c^2}{e^2B^2}\left[\frac{W_\perp^3}{3B}\hat{\perp}\cdot\nabla B + \frac{W_\parallel W_\perp^2}{8}\Big((\hat{\boldsymbol{\rho}}\cdot\nabla)\hat{\mathbf{b}}\cdot\hat{\boldsymbol{\rho}} - (\hat{\perp}\cdot\nabla)\hat{\mathbf{b}}\cdot\hat{\perp}\Big) + W_\parallel^2 W_\perp(\nabla\times\hat{\mathbf{b}})\cdot\hat{\boldsymbol{\rho}}\right]$$

$$S_1 \sim \mathcal{O}(\epsilon_B)$$

- **Guiding-center Hamiltonian**

$$H_{\rm gc} = \frac{1}{2}mW_\parallel^2 + \frac{1}{2}mW_\perp^2 + m\left(W_\parallel\mathcal{W}_\parallel + W_\perp\mathcal{W}_\perp\right) = \frac{1}{2}mW_\parallel^2 + \mu B$$

- **Guiding-center Phase-space Lagrangian up to the second order**

$$\gamma_{\rm gc} = \left(\frac{e}{c}\mathbf{A}(\mathbf{X}) + mW_\parallel\widehat{\mathbf{b}}(\mathbf{X}) - \frac{mc}{e}\mu\mathbf{R}\right)\cdot{\rm d}\mathbf{X} + \frac{mc}{e}\mu{\rm d}\Theta - H_{\rm gc}{\rm d}t + \mathcal{O}(\epsilon_B^2)$$

Guiding-center dynamical reduction

Homogeneous &
Non-homogeneous
magnetic field

Position and velocity shift
+
Gauge transformation:

Far from identity phase-space transformation

Larmor radius definition: **exact solution for SLAB** (homogeneous) magnetic field

Non-homogeneous magnetic field

Building a near-identity phase-space coordinate transformation with **gauge-transformations**

Goal: At the end of this step a symplectic part is free from gyroangle dependencies

↓

Poisson Bracket on the reduced phase space free from gyroangle dependencies **until the required order**

Non-homogeneous magnetic field

Final step:
Canonical Lie-transformation: on the Hamiltonian only (scalar function) use Poisson Bracket on the reduced phase space

Goal: remove all gyroangle dependencies from the Hamiltonian

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Gyrocenter dynamical reduction

- 1-form phase-space particle Lagrangian: **reinjection EM field fluctuations**

$$\gamma_{\rm pert} \equiv L_{\rm pert}\cdot {\rm d}t = \left(\frac{e}{c}\mathbf{A}(\mathbf{x}) + \epsilon_\delta \frac{e}{c}\mathbf{A}_1(\mathbf{x},t) + m\mathbf{v}\right)\cdot {\rm d}\mathbf{x} - H{\rm d}t$$

- **Perturbed Hamiltonian** $H = \frac{1}{2}m\mathbf{v}^2 + \epsilon_\delta e\phi_1(\mathbf{x},t)$

- **Gyrocenter Velocity shift** $\bar{\mathbf{v}} = \mathbf{v} + \epsilon_\delta \frac{e}{mc}\mathbf{A}_1(\mathbf{x},t)$

- Removes all the fluctuating fields from the symplectic part to the Hamiltonian part

$$\gamma_{\rm pert} = \left(\frac{e}{c}\mathbf{A}(\mathbf{x}) + m\bar{\mathbf{v}}\right)\cdot {\rm d}\mathbf{x} - H{\rm d}t$$

- Canonical Lie-transformations on the Hamiltonian part: Gyrocenter dynamical reduction

$$H = \frac{1}{2}m\bar{\mathbf{v}}^2 + \epsilon_\delta e\phi_1(\mathbf{x},t) - \epsilon_\delta\frac{e}{mc}\bar{\mathbf{v}}\cdot\mathbf{A}_1(\mathbf{x},t) + \epsilon_\delta^2\left(\frac{e}{mc}\right)^2\|\mathbf{A}_1(\mathbf{x},t)\|^2$$

First derivation in [Brizard 1989].

Gyrokinetic field theory: concept

Particles(GK) dynamical reduction
Goal: remove fastest scale of motion

[Littlejohn 1983]

Gauge transformation & Canonical Lie-Transform

Guiding-center

[Brizard 1989]

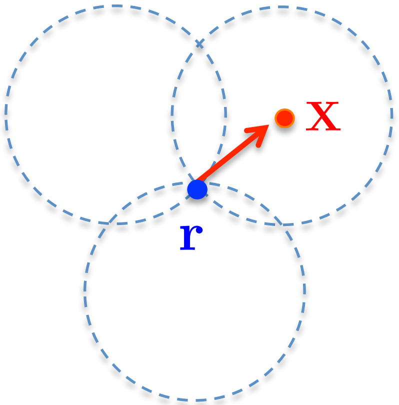
Canonical Lie-Transform

Gyrocenter

Variational principle:

Coupling fields & gyrocenters

Reduced GK Vlasov+ GK Maxwell equations



Bonus: Noether theorem for energy conservation diagnostics

- **Polarization effects: fields and particles are not evaluated at the same position anymore: polarization shifts in GK Ampére and Poisson equations**

Field theory guarantees consistency

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Gyrokinetic field theory: GENE & ORB5

- **Common framework for code models derivation:** *[**Sugama** Phys. Pl. 2000, **Brizard** PRL 2000]*

$$L = \sum_{\mathrm{s}} \int \mathrm{d}\Omega\ f(\mathbf{Z}_0, t_0)\ L_p\left(\mathbf{Z}[\mathrm{Z}_0, t_0], \dot{\mathbf{Z}}[\mathbf{Z}_0, t_0]; t\right) + \int \mathrm{d}V \frac{|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2}{8\pi}$$

- Phase-space volume $d\Omega = dV\,dW$
- **Field terms: option to couple with fluid model**

- Time-dependent: **GENE**
- Time-independent: **ORB5**

$$\mathbf{Z} = (\mathbf{X}, p_{\|}, \mu, \theta)\,; dW = \frac{2\pi}{m^2} B_{\|}^{**} dp_{\|} d\mu$$

$$\mathbf{Z} = (\mathbf{X}, p_z, \mu, \theta)\,; dW = \frac{2\pi}{m^2} B_{\|}^{*} dp_z d\mu$$

Goal: Coupling reduced particle dynamics with fields within the common mathematical structure

⬇

Getting consistently reduced set of Maxwell-Vlasov equations

$f(\mathbf{Z}_0, t_0)$ Distribution function of species “sp” at arbitrary initial time t_0

L_p Gyrocenter Lagrangian: reduced motion of a single particle

Gyrocenter Lagrangian: GENE & ORB5

$$L_p = \left(\frac{e}{c}\mathbf{A} + \left(\frac{e}{c}\epsilon_\delta A_{1\|} + mv_\|\right)\widehat{\mathbf{b}}\right)\cdot\dot{\mathbf{X}} + \frac{mc}{e}\mu\dot{\theta} - H$$

- **Parallel Symplectic representation: GENE**

$p_\| = mv_\|$ $\mathbf{Z} = \left(\mathbf{X}, p_\|, \mu, \theta\right)$

- Symplectic form: time dependent

$$\mathbf{B}^{**} = \boldsymbol{\nabla}\times\left(\mathbf{A} + \left[\epsilon_\delta A_{1\|} + \frac{c}{e}p_\|\right]\widehat{\mathbf{b}}\right)$$

- Characteristics with $\frac{\partial A_{1\|}}{\partial t}$

- **Hamiltonian representation: ORB5**

$p_z = mv_\| + \frac{e}{c}\epsilon_\delta A_{1\|}$ $\mathbf{Z} = \left(\mathbf{X}, p_z, \mu, \theta\right)$

- Symplectic form: time independent

$$\mathbf{B}^{*} = \boldsymbol{\nabla}\times\left(\mathbf{A} + \frac{c}{e}p_z\widehat{\mathbf{b}}\right)$$

[APS invited: Tronko, Bottino, Görler, Sonnendrücker, Told, Villard, PoP 2017]

Hamiltonian hierarchy: Theory & ORB5

- **Hamiltonian model defines polarization and magnetization in the field equations**
- **Any approximated model can be used: Padé, adiabatic electrons**

$$H = H_0 + \epsilon_\delta H_1 + \epsilon_\delta^2 H_2$$

$$\begin{aligned} H_0^{\mathrm{ORB5}} &= \frac{p_z^2}{2m} + \mu B \\ H_1^{\mathrm{ORB5}} &= -e\ \mathcal{J}_0^{\mathrm{gc}}\left(\psi_1^{\mathrm{ORB5}}\right) \\ \psi_1^{\mathrm{ORB5}} &= \phi_1 - (p_z/m)A_{1\|} \end{aligned}$$

- **Theory:** Hamiltonian **correspondance to Hahm's 1988 electrostatic model**

$$H_{2\mathrm{full}}^{\mathrm{THEORY}} = \frac{e^2}{2mc^2}\mathcal{J}_0^{\mathrm{gc}}\left(\mathbf{A}_{1\|}(\mathbf{X}+\boldsymbol{\rho}_0)^2\right) - \frac{e^2}{2B}\mathcal{J}_0^{\mathrm{gc}}\left(\frac{\partial}{\partial\mu}\widetilde{\psi}_1^{\mathrm{ORB5}}(\mathbf{X}+\boldsymbol{\rho}_0)^2\right)$$

Electromagnetic coupling between
GK Poisson and Ampère equations

- **ORB5 semi-electromagnetic**

$$H_2^{\mathrm{ORB5}} = \frac{e^2}{2mc^2}A_{1\|}(\mathbf{X})^2 + \frac{\mu}{2B}\left|\boldsymbol{\nabla}_\perp A_{1\|}(\mathbf{X})\right|^2 + \frac{1}{2}\frac{\mu}{B}A_{1\|}\boldsymbol{\nabla}_\perp^2 A_{1\|}(\mathbf{X}) - \frac{e^2}{2B}\mathcal{J}_0^{\mathrm{gc}}\left(\frac{\partial}{\partial\mu}\phi_1(\mathbf{X}+\boldsymbol{\rho}_0)^2\right)$$

Uncoupled GK Poisson and Ampère equations

Linearized Uncoupled GK Poisson and Ampère equations: ORB5

- **Polarization equation in strong form**

$$\frac{\delta L}{\delta \phi_1} \circ \phi_1 = 0 \quad \Rightarrow$$

$$\sum_{\mathrm{s}} \int \mathrm{d}\Omega \; f \; q_s \; \mathcal{J}_0^{\mathrm{gc}} \left(\phi_1\right) = \epsilon_\delta \sum_{s} \int \mathrm{d}\Omega \; f_C \; \frac{q_s^2}{B m_s} \frac{\partial}{\partial \mu} \left(\mathcal{J}_0^{\mathrm{gc}} \left(\phi_1^2\right) - \left[\mathcal{J}_0^{\mathrm{gc}} \left(\phi_1\right)\right]^2 \right)$$

- **Ampère's equation in strong form**

$$\frac{\delta L}{\delta A_{1\|}} \circ A_{1\|} = 0 \quad \Rightarrow$$

$$\epsilon_\delta \int \frac{dV}{4\pi} \; \left|\boldsymbol{\nabla}_\perp A_{1\|}\right|^2 = \sum_{\mathrm{s}} \int \mathrm{d}\Omega \; f \; \frac{p_z}{m_s} \mathcal{J}_0^{\mathrm{gc}} \left(A_{1\|}\right)$$

$$- \sum_{\mathrm{s} \neq e} \epsilon_\delta \int \mathrm{d}\Omega \; f_C \; \left(\frac{q_s^2}{m_s} A_{1\|}^2 + \frac{m_s \mu}{B} \left[A_{1\|} \boldsymbol{\nabla}_\perp^2 A_{1\|} + A_{1\|} \boldsymbol{\nabla}_\perp^2 A_{1\|} \right] \right)$$

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GK Vlasov equation: ORB5

- **Vlasov equation is reconstructed from the characteristics**

$$\frac{\delta L}{\delta \mathbf{Z}} = 0$$

$$\mathbf{Z} = (\mathbf{X}, p_z, \mu)$$

$$\dot{\mathbf{X}} = \frac{\partial(H_0 + \epsilon_\delta H_1)}{\partial p_z} \frac{\mathbf{B}^*}{B_\parallel^*} - \frac{c}{eB_\parallel^*} \widehat{\mathbf{b}} \times \boldsymbol{\nabla}(H_0 + \epsilon_\delta H_1)$$

$$\dot{p}_z = -\frac{\mathbf{B}^*}{B_\parallel^*} \cdot \boldsymbol{\nabla} \left(H_0 + \epsilon_\delta H_1\right)$$

$$\frac{d}{dt} f\left(\mathbf{Z}[\mathbf{Z}_0, t_0, t]; t\right) = \frac{\partial}{\partial t} f\left(\mathbf{Z}, t\right) + \frac{d\mathbf{Z}}{dt} \cdot \frac{\partial}{\partial \mathbf{Z}} f(\mathbf{Z}, t)$$

- **δf model requiers first order characteristics : only H_0 and H_1**
- **Full-f (nonlinear) model requires H_2 contributions in the characteristics**

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Part II: Numerical advantages of a consistent theory derivation

- **Ground code verification: Linear electromagnetic benchmark**
- Successfully accomplished; Results published

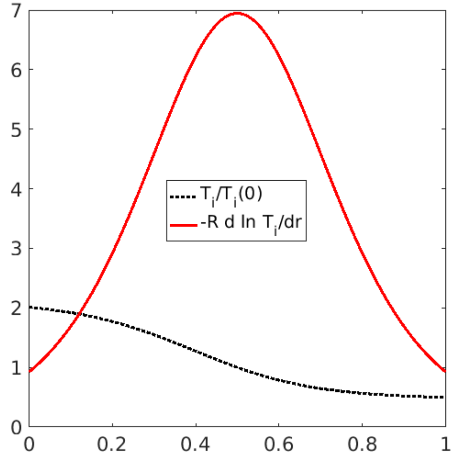
- **ORB5** New understanding of electromagnetic microinstabilities; new connections to fundamental GK derivation from the variational principle

Noether theorem ↔ **Powerbalance diagnostics**

Cyclone Base Case

- Common framework for benchmark: *[**Dimits,** Phys. Pl. 2000]*

  electrostatic simulations, adiabatic electrons



- The original discharge DIII-D:

  H- mode shot #81499 at t=4000 ms;

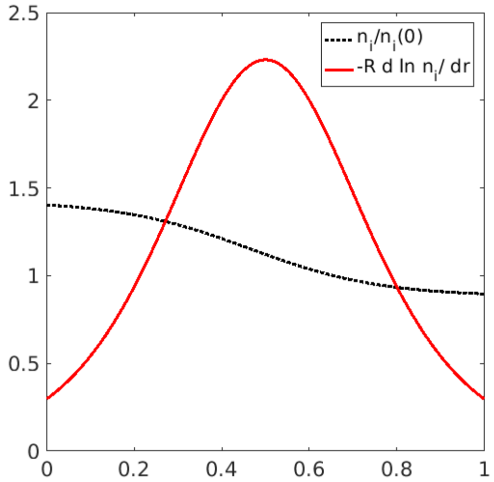
  flux tube label r=0.5a

$$q(r) = 0.86 - 0.16(r/a) + 2.52(r/a)^2$$

$$A(r) = A(r_0)\exp\left[-\kappa_A a \Delta A \tanh\left(\frac{r-r_0}{\Delta A a}\right)\right]$$

$$\Delta T_i = \Delta n = 0.3$$

$$\kappa_{T_i} = 6.96 \qquad \kappa_n = 2.23 \qquad T_e/T_i = 1$$



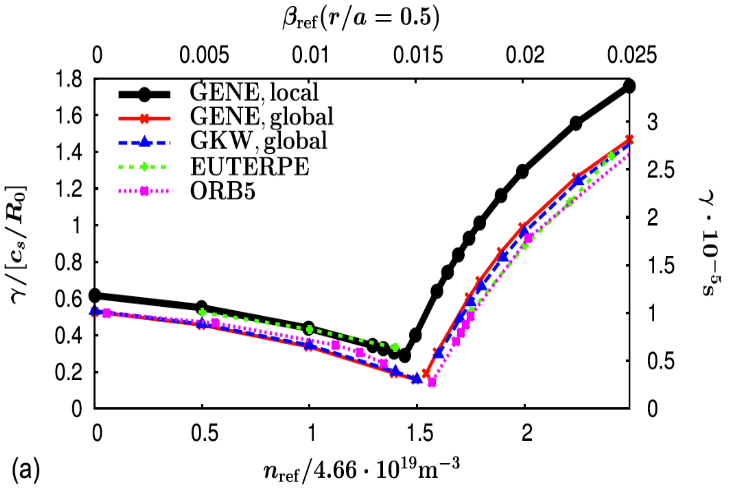
Linear electromagnetic β-scan: 5 codes

[Goerler, Tronko, Hornsby et al, PoP 2016]

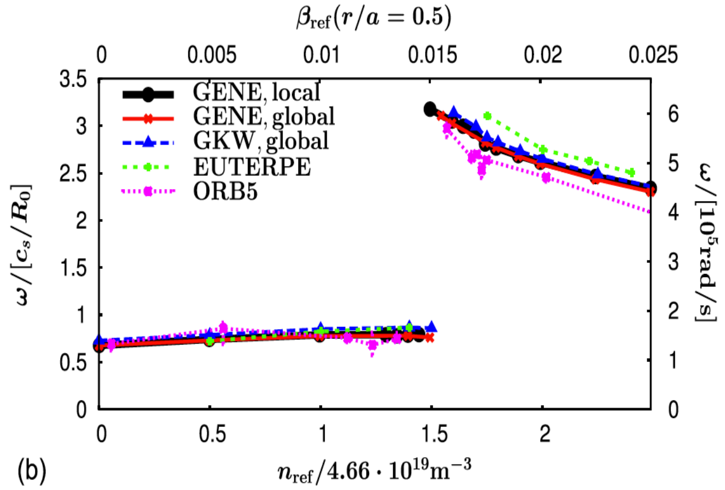
- Looking at one of the most unstable modes **n=19**
  - Successful comparison of 4 different codes (2xPIC and 2xEulerian)
  - All codes agree at the ITG/KBM transition
  - Threshold shifted comparing to flux-tube growth rate

Important for experiment

- ***Growth Rate and Frequency scan***



(a)



(b)

- **ORB5 code**

$$\mathrm{nptot}_{\mathrm{De}} = 8 \times 10^6$$

$$\mathrm{nptot}_{\mathrm{e}} = 16 \times 10^6$$

4 global codes 1 local (GENE)

Size of the system

$$\rho_* = \frac{\rho_L}{a} = \frac{1}{180}$$

Electromagnetic Powerbalance ORB5

- **Noether theorem**

$$\mathcal{E} = \mathcal{E}_k + \mathcal{E}_F$$

$$\mathcal{E}_k = \sum_{\mathrm{s}} \int \mathrm{d}\Omega\; f \left(\frac{p_z^2}{2m_s} + \mu B \right)$$

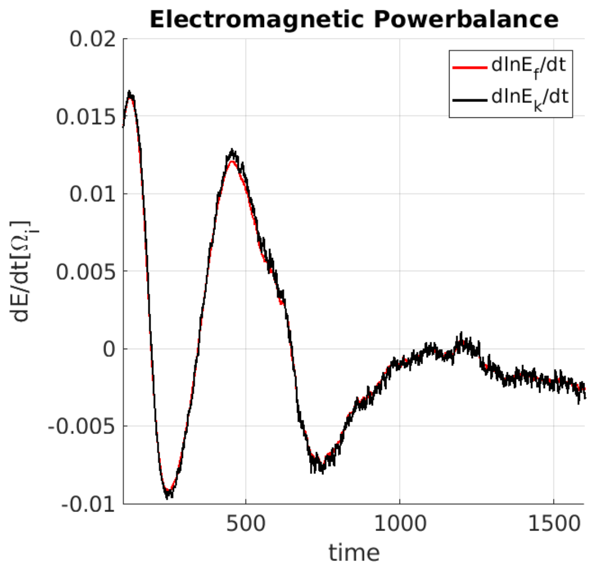
Particles energy

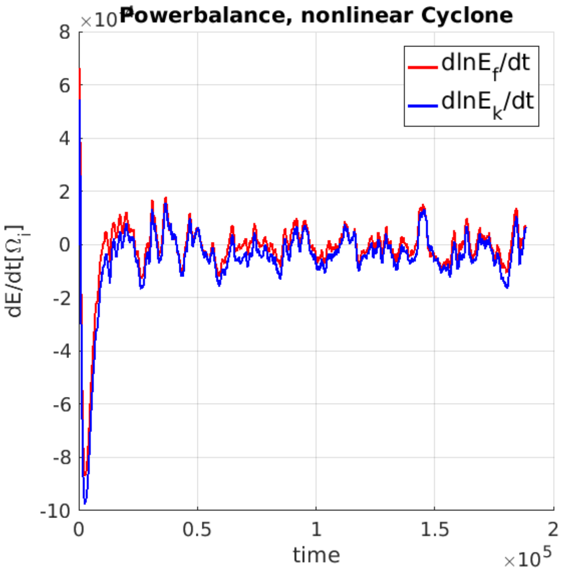
$$\mathcal{E}_F = \frac{1}{2} \sum_{\mathrm{s}} \int \mathrm{d}\Omega\; f \left(\phi_1 - \frac{e p_z}{mc} A_{1\|} \right)$$

Fields energy

- **Verification of energy conservation in the simulations**

$$\boxed{\frac{d \ln \mathcal{E}_k}{dt} = - \frac{d \ln \mathcal{E}_F}{dt}}$$





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Open questions and further developments

- **GK codes: significant development: electromagnetic implementations**
  - **Electrostatic gyrokinetic implementations :** theory & simulations: **well established for core of Tokamak**
  - **Global electromagnetic gyrokinetic implementations:**
    - Next level of complexity: Alfvén physics
    - A lot of freedom for approximations (Poisson and Ampère equations)
    - **Different codes implement different version of GK equations**
  - **Need to question existing orderings**
    - **Comparing with experiments**
    - **From the core to the edge of devices: very different physical properties**
    - **Exploring model validity in new regimes**
    - **New magnetic geometries (Stellamak: under construction)**

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