

HAMILTONIAN DESCRIPTION OF PLASMAS AND OTHER MODELS OF MATTER

PHIL MORRISON

Survey of physical systems
Applied aspect of this workshop.

Hamiltonians are the basis of physical theories - Why?

- Beautiful
ancient idea - set derivative of something equal to zero
teleology - God doesn't do anything in vain
- Symmetries & Conservation Laws easily seen
- Generality
- Approximations - perturbation theory is easier since the dynamics are contained in a single function
- finite dim. Hamiltonian theory suggests results for ∞ -dim systems
what happens if there's a continuous derivative?
- techniques to check for stability
- numerical methods that preserve structure
- gateway to statistical mechanics

Action Principle

Hamilton's Principle
originally done for life

Procedure:

- describe configuration space
- write a Lagrangian
- construct an action functional
- extremal path \Rightarrow Lagrange's equations

Variation over Paths

} described this morning

Hamilton's Equations

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

phase space coordinates: $z = (q, p)$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\} \quad \text{where } (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

J_c = Poisson tensor

the symplectic 2-form ω satisfies:

$$\omega_{\alpha\beta}^c J_c^{\beta\gamma} = \delta_\alpha^\gamma$$

\uparrow $N \times N$ matrix of all zeroes
 \uparrow $N \times N$ identity matrix

Natural Hamiltonian Systems

Separates into a kinetic & potential energy.

kinetic: $\frac{1}{2} \sum_{ij} m_{ij}^{-1}(q) p_i p_j$; m_{ij}^{-1} = positive definite mass matrix

potential: $V(q_1, \dots, q_n)$ no momentum dependence.

Ex: masses & springs; pendula; particle in potential well;

N-body problem

sign of interaction \rightarrow gravity vs. electrostatic

Unnatural Hamiltonian Systems

Bad name since many physical systems are "unnatural"

- Interaction of point vortices in a plane

$$H = c \sum_{i,j=1}^N \kappa_i \kappa_j \log((x_i - x_j)^2 + (y_i - y_j)^2)$$

- Charged particle in an electromagnetic field.

Hamiltonian contains a $\vec{p} \cdot \vec{A}$ term.

- Predator-prey systems

- Dye flowing in a 2-dim fluid (incompressible)

Hamiltonian is the stream function

- Integral fields of a magnetic field

- Chaotic advection \rightarrow non-twist - different routes for chaos experiment

↑ Sumner: "I think this is a KAM torus"

- Particle in a magnetic field.

gyroradius important since we can trap particles along a field line

- B lines as a Hamiltonian system

basic idea for plasma confinement

- Tokamak field

two fields drawn together

helical field lines inside a torus

if done just right, we get nested invariant tori

"time" is the toroidal angle. τ

if no $\tau \rightarrow$ integrable system

if $\tau \rightarrow 1/2$ d.o.f system.

* *

* * Surface of Section

Cut through torus

with symmetry (no z),

invariant tori

typical curve densely covers torus (unless rational)

add z -dependence makes system more complicated pictures are for standard map.

Early Symplectic Maps

Standard Map

studied by physicists to understand magnetic confinement

Standard Non-twist Map

written down for Swinney's fluid experiment

VB Drift

can't have strong B everywhere finite device \rightarrow drifts

Gyromotion / Guiding Center

merry-go-round as an airplane we care more about motion of plane.

describe motion of guiding center instead of particles

Noncanonical Hamiltonian System

usual geometry

J_c is the one we saw before: $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

noncanonical: defined in terms of the bracket.

Lie algebra doesn't require Leibniz

bracket has a different J

Darboux: $\det J \neq 0 \Rightarrow$

local transform to canonical coords

Lie: $\det J = 0$

Casimirs - invariants build into the phase space

due to degeneracy of Poisson brackets

(Lie's distinguished functions) might be a better name.

Associated with any finite-dim.

Lie algebra will give you a bracket

∞ -dim version: J is an operator

Lie-P. geometry

coord-free way of talking about this.

Classified Lie algebras

Bianchi cosmologies

What is the foliation by

Casimirs for each type?

black - A class - algebraic varieties

red - class B - singularities in surfaces

- Class A {
 - Hershey
 - Kide Vortex - exact, time-dep solution (time-dep) of Euler's eq
 - rigid sphere

- Class B {
 - rattleback (slowtime)
 - chirality
 - J has singularity
 - equilibrium on the singularity, despite smooth vector field.

① Magnetic field line flows

some restrictions

what if B vanishes?

is this a class B system?

② Rattleback is integrable

3-dim, foliated by Casimirs

1-dof. on the leaves

③ Are symplectic forms understood

for all types/class B? No

④ How are ^{magnetic} equilibria affected by

moving particles?

usually solve MHD for an equilibrium.

then look at particle motion about

MHD equilibrium

much more complicated

rank-changing events:

chirality

may not be able to find

a eq. by varying flam & Cas

"Cas. deficit problem"

unt research Casimirs

Infinite-Dimensional Systems (Field Theories)

dynamics of matter
PDE / PDIE

last time, everything could be made rigorous today, no.

scalar product - pairing (more precise)
integration variable - a.k.a. label

you can get around using distributions - we're not since it's not rigorous anyway.

Poisson tensor replaced by operator. Goal to understand this operator.
Challenge is often to prove Jacobi identity

Lie-algebra infinite dimensional

group theory for this is (currently) nice, picturesque magery - but not mathematical.

H-M = Hasegawa-Mima (I should expand acronyms)

for any matter model

you can point to dissipative term - has a ~~time~~ measured parameter (look up)
if you remove this term, what should remain is a Hamiltonian field theory

we will focus on theories with interesting \mathcal{J}

Ex - stick this Hamiltonian in to get the KdV eqn.

\mathcal{J}_B - in a plasma, the total charge is zero. stellar dynamics - sign changes, no background.

Vlasov-Poisson system

actually older than Vlasov - collisionless Boltzmann / Gene's Eqn.

if ϕ were a given function of (x,t) , we could solve easily by method of characteristics
existence theory in 1d in 1960's
global existence theory not until 1989.

Commonality

many systems have the same structure / Poisson bracket - only \mathcal{H} different

QG = Quasigeostrophy

VP & 2d Euler

both quadratically nonlinear, energy is quadratic
has same Casimirs, other properties

expectation of particle bracket gives you the field bracket.

MHD - magnetohydrodynamics - coined by Alfvén - 1950's

first 2 eqn, excluding $\vec{J} \times \vec{B}$ is Euler's Eqn.

Ohm's law: $V = IR$

replace $E \rightarrow E + v \times B$

$E = \eta J$ (divide by volume)

electric field in the frame of the fluid

ideal case: no resistivity: $\eta = 0$: replace E with $- \vec{v} \times B$

one eqn missing: $J = \nabla \times B$ - neglect $\frac{\partial E}{\partial t}$.

to find pressure, use thermodynamics

$$T = \frac{\partial U}{\partial S}$$

no existence theorem.

$\vec{M} = \rho \vec{v}$ momentum density

bracket is linear in dynamical variables

can directly prove Jacobi - a week.

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Casimir invariants - null space of Poisson operator - commute with any functional
helicities were already known - magnetic & cross
can't count constants to determine # Casimirs
have to use functional analysis/operator theory to understand

XMHD - extended MHD

Ohm's law gets a bunch of junk on the RHS - stick that in Faraday's law
also adds a term to momentum

Hall term - ion physics } MHD has ion physics
- electron physics }

can rewrite in terms of dimensionless numbers

the bracket for XMHD is the bracket of MHD + max wags - not even Lie-Poisson
actually reduces to MHD bracket with the appropriate transformation of fields
in these variables, XMHD reduces to Lie dragging of two Z -forms
you get generalize helicities out of this

Maxwell-Vlasov

all of Maxwell's Eqns - coupled to Vlasov

reduce to point particles - $f \rightarrow \sum \delta(\dots)$

reduce to fluids - $f(x, v, t) = f(x, v(x, t), t)$

numerical challenge - find a discrete version this - symplectic integrator

Casimirs give you constraint Maxwell's Eqns

Where does this structure come from?

You can first construct a canonical particle theory, then do a Hamiltonian reduction.

$(q, p) \mapsto w(q, p)$ no a coord change - fewer w than (q, p)

something special might happen - closure

\rightarrow subalgebra.

the Poisson bracket for this is non canonical

if you're lucky, Hamiltonian also only depends on w .

this doesn't give you as much information, but it is a simpler problem

if only almost true, we could do perturbation theory.

Kivshin point vortices - ODE from Euler's Eqns

Contour Dynamics - vorticity constant inside a patch patch

look at the dynamics of the boundary of the patch.

Further reduction \rightarrow Type VIII

this are special initial conditions - not asymptotics / approx

Asymptotic Reductions

possible, but this is an underdeveloped area.

Where does the funny bracket for MHD come from? A reduction.

description of Eulerian vs. Lagrangian pictures

$q_t: D \rightarrow D$ we don't have theorems saying what this is, so we assume what we need.

Q Why don't people use contour dynamics more?

Dritschel, et. have developed a numerical method for it

challenge - you get filaments - have to snip them off