2A

HAMILTONIAN DESCRIPTION OF PLASMAS AND OTHER MODELS OF MATTER

PHIL MORRISON

Survey of physical systems

Applied aspect of this workshop.

Hamiltonians are the basis of physical theories - Why?

- Beautiful
  - ancient idea - set derivative of something equal to zero
  - teleology - God doesn't do anything in vain
- Symmetries & Conservation laws easily seen
- Generality
- Approximations - perturbation theory is easier since the dynamics are contained in a single function
- finite dim. Hamiltonian theory suggests results for ∞ - dim systems
  - what happens if there's a continuous derivative?
- techniques to check for stability
- numerical methods that preserve structure
- gateway to statistical mechanics

Action Principle

Hamilton's Principle

originally done for life

Procedure:

describe configuration space

write a Lagrangian

construct an action functional

extremal path $\Rightarrow$ Lagrange's equations

Variation over Paths

Hamilton's Equations

$\Big\}$ described this morning

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

phase space coordinates: $z = (q,p)$

$$\dot{z}^\alpha = J_c^{\alpha\beta} \frac{\partial H}{\partial z^\beta} = \{z^\alpha, H\} \quad \text{where } (J_c^{\alpha\beta}) = \begin{pmatrix} O_N & I_N \\ -I_N & O_N \end{pmatrix}$$

O_N: N×N matrix of all zeroes

I_N: N×N identity matrix

J_c = Poisson tensor

the symplectic 2-form ω satisfies:

$$\omega^c_{\alpha\beta} J_c^{\beta\gamma} = \delta_\alpha^\gamma$$

Natural Hamiltonian Systems

Separates into a kinetic & potential energy.

kinetic: $\frac{1}{2}\sum_{ij} m_{ij}^{-1}(q)\ p_i p_j$; m_{ij} = positive definite mass matrix

potential: $V(q_1, \dots, q_N)$ no momentum dependence.

Ex: masses & springs; pendula; particle in potential well; N-body problem

sign of interaction → gravity vs. electrostatic

Unnatural Hamiltonian Systems

Bad name since many physical systems are "unnatural"

- Interaction of point vortices in a plane

$$H = c\sum_{i,j=1}^{N} \kappa_i \kappa_j \log\left((x_i - x_j)^2 + (y_i - y_j)^2\right)$$

- Charged particle in an electromagnetic field.
  Hamiltonian contains a $\vec{p}\cdot\vec{A}$ term.
- Predator-prey systems
- Dye flowing in a 2-dim fluid (incompressible)
  Hamiltonian is the stream function
- Integral fields of a magnetic field

- Chaotic advection → nontwist - different routes for chaos
  experiment
  Swinney: "I think this is a KAM torus"
- Particle in a magnetic field.
  gyroradius important since we can trap particles along a field line
- B lines as a Hamiltonian system
  basic idea for plasma confinement
- Tokamak field
  two fields drawn together
  helitical field lines inside a torus
  if done just right, we get nested invariant tori

"time" is the toroidal angle. z

if no z → integrable system

if z → 1½ d.o.f. system.

* *

* *

Surface of Section

Cut through torus

with symmetry (no z),

invariant tori

typical curve densely covers atoms (unless rational)

add z-dependence makes system more complicated

pictures are for standard map.

Early Symplectic Maps

Standard Map

studied by physicists to understand magnetic confinement

Standard Nontwist Map

written down for Swinney's fluid experiment

∇B Drift

can't have strong B everywhere

finite device → drifts

Gyromotion / Guiding Center

merry-go-round on an airplane

we care more about motion of plane.

describe motion of guiding center instead of particles

Noncanonical Hamiltonian System

usual geometry

J_c is the one we saw before: $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

noncanonical : defined in terms of the bracket.

Lie algebra doesn't require Leibniz

bracket has a different J

Darboux : $\det J \neq 0 \Rightarrow$
local transform to canonical coords

Lie : $\det J = 0$

Casimirs - invariants build into the phase space

due to degeneracy of Poisson brackets

(Lie's distinguished functions) might be a better name.

Associated with any finite-dim. Lie algebra will give you a bracket

∞-dim version : J is an operator

Lie-P. geometry
coord-free way of talking about this.

Classified Lie algebras
Bianchi cosmologies

What is the foliation by Casimirs for each type?

black - A class - algebraic varieties

red - class B - singularities in surfaces

class A
- Heisenberg
- Kida Vortex - exact, time-dep solution (time-dep) of Euler's eq
- rigid sphere

class B
rattle back (showtime)
└ chirality
J has singularity
equilibrium on the singularity despite smooth vector field.

rank-changing events:
chirality
may not be able to find a eq. by varying Ham & Cas
"Cas. deficit problem"
not enough Casimirs

Q Magnetic field line flows
some restrictions
what if B vanishes?
is this a class B system?

Q Rattleback is integrable
3-dim, foliated by Casimirs
1-dof. on the leaves

Q Are symplectic forms understood for all types / class B? No

Q How are magnetic equilibria affected by moving particles?
usually solve MHD for an equilibrium.
then look at particle motion about MHD equilibrium
much more complicated.

Infinite - Dimensional Systems (Field theories)

dynamics of matter
PDE / PDIE

last time, everything could be made rigorous
today, no.

scalar product - pairing (more precise)
integration variable - a.k.a. label
you can get around using distributions - we're not since it's not rigorous anyway.

Poisson tensor replaced by operator. Goal to understand this operator.
Challenge is often to prove Jacobi identity

Lie - algebra infinite dimensional
group theory for this is (currently) nice, picturesque imagery - but not mathematical.

H-M = Hasegawa - Mima (I should expand acronyms)
for any matter model
you can point to dissipative term - has a ~~[illegible]~~ measured parameter (look up)
if you remove this term, what should remain is a Hamiltonian field theory

we will focus on theories with interesting J
Ex - stick this Hamiltonian in to get the KdV eqtn.

ρ_B - in a plasma, the total charge is zero.
stellar dynamics - sign changes, no background.

Vlasov - Poisson system
actually older than Vlasov - collisionless Boltzmann / Gene's Eqtn.
if Φ were a given function of (x,t), we could solve easily by method of characteristics
existence theory in 1d in 1960's
global existence theory not until 1989.

Commonality
many systems have the same structure / Poisson bracket - only H different
QG = quasigeostrophy
VP & 2d Euler — both quadratically nonlinear, energy is quadratic
has same Casimirs, other properties
expectation of particle bracket gives you the field bracket.

MHD - magnetohydrodynamics - coined by Alfven - 1950's
first 2 eqtn, excluding $\vec{J}\times\vec{B}$ is Euler's Eqtn.
Ohm's Law: $V = IR$
$E = \eta J$ (divide by volume)
replace $E \to E + v\times B$
electric field in the frame of the fluid
ideal case: no resistivity: $\eta = 0$: replace E with $-v\times B$
one eqtn missing: $J = \nabla \times B$ - neglect $\frac{\partial E}{\partial t}$.
to find pressure, use thermodynamics
$T = \frac{\partial U}{\partial S}$
no existence theorem.
$\vec{M} = \rho\vec{v}$ momentum density
bracket is linear in dynamical variables
can directly prove Jacobi - a week.

Casimir invariants — null space of Poisson operator - commute with any functional
helicities were already known - magnetic & cross
can't count covariant to determine # Casimirs
have to use functional analysis/operator theory to understand

XMHD — extended MHD

Ohm's law gets a bunch of junk on the RHS — stick that in Faraday's law
also adds a term to momentum

Hall term - ion physics
- electron physics } MHD has one physics

can rewrite in terms of dimensionless numbers

the bracket for XMHD is the bracket of MHD + more mess - not even Lie-Poisson
actually reduces to MHD bracket with the appropriate transformation of fields
in these variables, XMHD reduces to Lie dragging of two 2-forms
you get generalize helicities out of this

Maxwell-Vlasov

all of Maxwell's Eqtns - coupled to Vlasov
reduce to point particles - $f = \sum \delta(\cdots)$
reduce to fluids - $f(x,v,t) = f(x, v(x,t), t)$
numerical challenge - find a discrete version this - symplectic integrator
Casimirs give you constraint Maxwell's Eqtns

Where does this structure come from?
You can first construct a canonical particle theory, then do a Hamiltonian reduction.
$(q,p) \mapsto w(q,p)$ no a coord change - fewer w than (q,p)
something special might happen - closure
↳ subalgebra
the Poisson bracket for this is noncanonical
if you're lucky, Hamiltonian also only depends on w.
this doesn't give you as much information, but it is a simpler problem
if only almost true, we could do perturbation theory.

Kirchhoff point vortices - ODE from Euler's Eqtns
Contour Dynamics - vorticity constant inside a ~~path~~ patch
look at the dynamics of the boundary of the patch.
further reduction → Type VIII

this are special initial conditions - not asymptotics / approx
Asymptotic Reductions
possible, but this is an underdeveloped area.

Where does the funny bracket for MHD come from? A reduction.

description of Eulerian vs. Lagrangian pictures

$g_t : D \to D$ we don't have theorems saying what this is, so we assume what we need.

Q Why don't people use contour dynamics more?

Dritschel, etc. have developed a numerical method for it

challenge - you get filaments - have to snip them off