

INTRODUCTION TO WEAK KAM THEORY

MARIE-CLAUDE ARNAUD

Concerns weak solutions of Hamilton-Jacobi equations

I. Definitions and Assumptions

Notation: $\mathbb{R}^n \ni x, t \mapsto x(t)$, \dot{x} deriv with respect to t
 \ddot{x} 2nd deriv with respect to t
 $\|\cdot\|$ usual Euclidean norm
 ∇ gradient

Historical Example Celestial Mechanics

Newton's equations:

$$m \ddot{x} = \nabla U(x)$$

U = potential function

$$U: \mathbb{R}^n \rightarrow \mathbb{R}$$

m mass

If $L(x, v) = \frac{1}{2} m \|v\|^2 + U(x)$ is the Lagrangian,
we can rewrite Newton's equations as:

Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v}(x, \dot{x}) \right) = \frac{\partial L}{\partial x}(x, \dot{x}) \quad (EL)$$

Subexample Free particle, $U=0$

$$m \ddot{x} = 0, \quad x(t) = x_0 + t v_0$$

\uparrow initial position \leftarrow initial speed

look at both (position, speed)

$$\begin{cases} \dot{x} = v \\ m \dot{v} = \nabla U(x) \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{x} = v \\ \frac{d}{dt} \left(\frac{\partial L}{\partial v}(x, v) \right) = \frac{\partial L}{\partial x}(x, v) \end{cases}$$

(2nd order diff eq \rightarrow needs two initial conditions)

we will need to add some additional hypotheses on L - soon.

Flow "follow the solution with initial (position, speed) = (x, v) "

$$\phi_t^L: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n, \quad \phi_t^L(x, v) = (x(t), v(t))$$

map of position & speed to itself

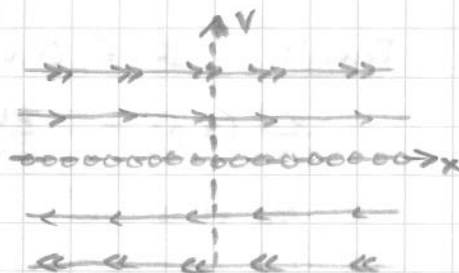
Subex $\phi_t^L(x, v) = (x + tv, v)$

We've partitioned (x, v) space into subsets where $v = \text{constant}$

Remark Invariant by the flow subsets:

$$\{(x, v), v = v_0\}$$

For $U=0$



Question of Weak KAM Theory

What happens if $N > 1$ and $V \neq 0$?

In general, you don't have an invariant, smooth foliation as for $V=0$.

→ in a x -periodic case, perturbative, apply (strong KAM)

→ what happens in general?

Problem of Underdetermination

we want the Lagrangian to determine the flow → add hypotheses

Def $L: \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ C^2 = Lagrangian

• L is non-degenerate if everywhere, $\det \left(\frac{\partial^2 L}{\partial v^2} \right) \neq 0$.

• the Legendre transform is the C^2 map

$$\begin{aligned} \mathcal{L}_L: \mathbb{R}^N \times \mathbb{R}^N &\rightarrow \mathbb{R}^N \times (\mathbb{R}^N)^* \\ (x, v) &\mapsto \left(x, \frac{\partial L}{\partial v}(x, v) \right) \end{aligned}$$

Remark this is a local diffeomorphism when L is non-degenerate

Remarks

when L is non-degenerate, (EL) can be rewritten as follows:

$$\frac{\partial^2 L}{\partial v^2}(x(t), \dot{x}(t)) \cdot \ddot{x}(t) = \frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{\partial^2 L}{\partial x \partial v}(x(t), \dot{x}(t)) \cdot \dot{x}(t)$$

no underdetermination

• if \mathcal{L}_L is a C^2 diffeomorphism onto $\mathbb{R}^N \times (\mathbb{R}^N)^*$

define new coordinates:

$$(x, p) = \mathcal{L}_L(x, v)$$

define a new flow $\varphi_t = \mathcal{L}_L \circ \beta_t \circ \mathcal{L}_L^{-1}$

define the Hamiltonian $H: \mathbb{R}^N \times (\mathbb{R}^N)^* \rightarrow \mathbb{R}$

$$H(x, p) = p \cdot v - L(x, v) \text{ where } (x, p) = \mathcal{L}_L(x, v)$$

in coordinates (x, p) , E-L equations become Hamilton's equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p) \\ \dot{p} = -\frac{\partial H}{\partial x}(x, p) \end{cases} \quad \left(\begin{array}{l} \text{a priori, it appears as} \\ \text{though } H \text{ is } C^1 \end{array} \right)$$

Remark If $\mathcal{L}_L(x, v) = (x, p)$,

$$\frac{\partial H}{\partial x}(x, p) = -\frac{\partial L}{\partial x}(x, v), \quad \frac{\partial H}{\partial p}(x, p) = v, \quad \frac{\partial L}{\partial v}(x, v) = p \leftarrow \text{Everything here is } C^2.$$

So H is C^2

Proposition Assume that $L: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 ,

is a Lagrangian that is

- C^2 convex in the vertical direction, i.e. $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite
- L is superlinear in the vertical direction, i.e. $\lim_{\|v\| \rightarrow +\infty} \frac{L(x, v)}{\|v\|} = +\infty$

then \mathcal{L}_L is a C^2 diffeomorphism onto $\mathbb{R}^n \times (\mathbb{R}^n)^*$

the Hamiltonian satisfies the same hypotheses:

C^2 convex and superlinear in the vertical direction

moreover (1) Young Inequality

$\forall (x, v, p)$ (don't assume $\mathcal{L}_L(x, v) = (x, p)$)

$$L(x, v) + H(x, p) \geq p \cdot v$$

with $=$ iff $\mathcal{L}_L(x, v) = (x, p)$

$$(2) H(x, p) = \max_{v \in \mathbb{R}^n} (p \cdot v - L(x, v))$$

Example $L(x, v) = \frac{m}{2} \|v\|^2 + U(x)$

$$\mathcal{L}_L(x, v) = (x, mv) \quad \text{this is a diffeomorphism}$$

$$H(x, p) = \underbrace{\frac{\|p\|^2}{2m}}_{\text{Kinetic Energy}} - \underbrace{U(x)}_{\text{Potential Energy}}$$

Def L (or H) are Tonelli

if C^2 -convexity and superlinearity in the vertical direction

II. Lagrangian Action

$L: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 , Tonelli

(uses minimization in a variational setting)

Def If $\gamma: [a, b] \rightarrow \mathbb{R}^n$ C^0 + piecewise C^1

← paths in position space

• its action is $A_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma})$

• variation of γ

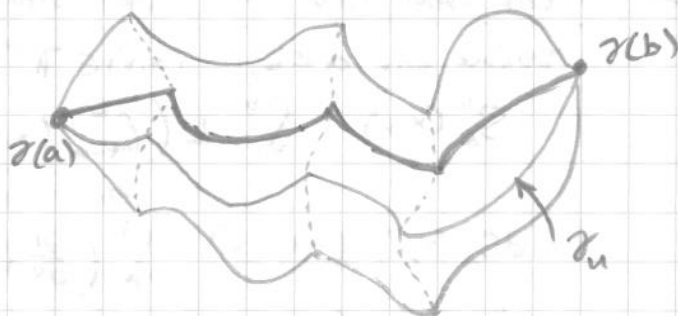
$$I: (-\varepsilon, \varepsilon) \times [a, b] \rightarrow \mathbb{R}^n$$

$$(u, t) \mapsto \gamma_u(t)$$

$$\bullet \forall u, \gamma_u|_{[a, b]} = \gamma|_{[a, b]} \quad \bullet \gamma_0 = \gamma$$

• I and $\frac{\partial I}{\partial u}$ are C^0

• \exists a subdivision $a = \tau_0 < \dots < \tau_k = b$ such that on $(-\varepsilon, \varepsilon) \times [\tau_i, \tau_{i+1}]$,



$$\frac{\partial^2 \Gamma}{\partial u^2} = \frac{\partial^2 \Gamma}{\partial t \partial u} \text{ is } C^0$$

For example, $\gamma_u(t) = \gamma(t) + u \gamma_1(t)$

$$\gamma_1(a) = \gamma_1(b) = 0, \quad \gamma_1 \text{ is } C^0 \text{ and piecewise } C^1$$

γ is an extremal of A_L if

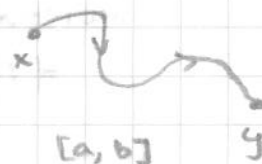
$$\text{for every variation } \frac{\partial}{\partial u} (A_L(\gamma_u))_{u=0} = 0$$

Proposition extremal \Leftrightarrow solution of E-L equation

Assumption \mathbb{Z}^N -periodicity (N-torus)

$$L(x+k, v) = L(x, v) \quad \forall k \in \mathbb{Z}^N$$

\Rightarrow existence of minimizers of the action with fixed ends and they are extremals.



Q What would be different on a sphere instead of a torus?

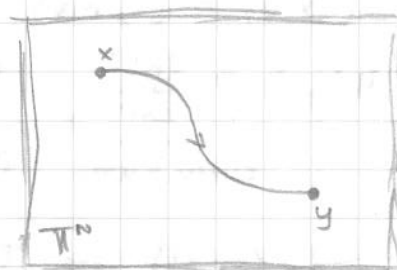
From now on, we assume the Lagrangian is Tonelli & periodic in the x-variable.

Then

For every $x, y \in \mathbb{T}^N$ and $a < b \in \mathbb{R}$

there exists an absolutely continuous path
(we no longer have piecewise C^1)

$\gamma: [a, b] \rightarrow \mathbb{T}^N$ such that $\gamma(a) = x, \gamma(b) = y$
which minimizes the Lagrangian action.



$\gamma: [a, b] \rightarrow \mathbb{T}^N$
time interval

Weierstrass satisfies E-L and is C^2 .

Lagrangian action: $a_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma}) dt$

Define a new action between two points for a given time:

$$a(x, y; t) = \min_{\substack{\gamma(a)=x \\ \gamma(b)=y}} a_L(\gamma) \quad \text{for } t > 0, x, y \in \mathbb{T}^N$$

III WEAK KAM THM - LAGRANGIAN POINT OF VIEW

Define for $a < b$, $(x, v) \in \mathbb{T}^N \times \mathbb{R}^N$ an action:

$$S_a^b(x, v) = \int_a^b L(\underbrace{\gamma_{t-a}^L(x, v)}_{\text{this is some } x(t), \dot{x}(t) \text{ where } x \text{ solves E-L}}) dt$$

$$dS_a^b(x, v) (\delta x, \delta v) = \int_a^b \left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) \delta x(t) + \frac{\partial L}{\partial v}(x(t), \dot{x}(t)) \delta \dot{x}(t) \right) dt$$

integrate by parts

$$\left(\text{where } (\delta x(t), \delta \dot{x}(t)) = D \gamma_{t-a}^L(x, v) (\delta x, \delta v) \right)$$

$$= \int_a^b \frac{\partial L}{\partial v}(x(t), \dot{x}(t)) \delta x(t) dt + \int_a^b \left(\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt} \left(\frac{\partial L}{\partial v}(x, \dot{x}) \right) \right) \delta x(t) dt$$

$$= p(b) \delta x(b) - p(a) \delta x(a)$$

$$\text{if } (x(t), p(t)) = \mathcal{L}_L(x(t), \dot{x}(t)) = \left(x(t), \frac{\partial \mathcal{L}}{\partial v}(x(t), \dot{x}(t)) \right)$$

$$p(b) \delta x(b) = \underbrace{p(a)}_{d_u a(x(a))} \delta x(a) + d S_a^b(x, v) (\delta x, \delta v)$$

if the image by the flow of the graph ($d u_a$) is also a graph, it is the graph of $d u_b$.

$$\text{with } u_b(x) = u_a(x) + S_a^b(x, v) + C$$

Def The Lax-Oleinik semigroup

$T^t C^0(\mathbb{T}^n, \mathbb{R}) \ni$ is defined by

$$T^t u(x_0) = \min_{x \in \mathbb{T}^n} (u(x) + a(x, x_0; t))$$

This describes how a graph u is changed by the flow.

Properties

- semigroup (only positive t)
- nonexpansiveness
- order preserving
- addition of constant not important
- continuity with respect to t

Thm Weak KAM, part 1

there is a fixed point of the weak semi-group

Idea of Proof

addition of constant not important
work in $C^0(\mathbb{T}^n, \mathbb{R}) / \mathbb{R}$

$$\|u\| = \min_{a \in \mathbb{R}} \|u - a\|_\infty$$

non expansive \rightarrow contracting

prove fixed point for each t , then prove a fixed curve for all t .

the fixed point is called the weak KAM solution

defined up to a constant C

Thm Weak KAM, part 2

• Dominator Property:

$$u(x) = \min_{\gamma(t)=x} \left(u(\gamma(0)) + \int_0^t (L(\gamma, \dot{\gamma}) + c) \right)$$

$$\Rightarrow \text{if } \gamma: [0, t] \rightarrow \mathbb{T}^n, \text{ then } u(\gamma(t)) - u(\gamma(0)) + \int_0^t (L(\gamma, \dot{\gamma}) + c)$$

• We have a solution on the negative interval

solution is γ_k

Solution may not be unique

Solution of E-L since it is minimizing on $[-\infty, 0]$

- c is unique

$c = c[0]$ is Mañé critical value

$$c = -\inf_{\mu} \int L d\mu, \quad \mu \text{ is a Borel invariant probability by } \beta_t^L$$

Idea of Proof

$$u(\pi \circ \beta_t^+(x, v)) - u(\pi(x, v)) \leq \int_0^t (L(\beta^s(x, v)) + c) ds$$

↑
projection on \mathbb{T}^N -space

integrate with respect to μ

$$0 \leq \int_0^t (\int L + c) d\mu$$

$$\Rightarrow -t \int L d\mu \leq ct$$

To get equality, work with ∂_x
(calibrated curve) from the second
part of the thm.

③ Why is this called the weak KAM thm?

KAM thm:

when you perturb an integrable Lagrangian system,
some tori break & some survive. — these are KAM tori

the KAM tori will be graphs of u . — when they exist

IV HAMILTONIAN POINT OF VIEW

If the weak KAM solution u_- is differentiable at $x \in \mathbb{T}^N$, then

• it's in a Hamiltonian level set $H(x, du_-(x)) = c$

• the calibrated curve satisfies the dynamics $du_-(x) = \frac{\partial L}{\partial v}(x, \dot{\gamma}_x(0))$

Hamilton - Jacobi Equation

$$H(x, du(x)) = c \text{ satisfies HJE}$$

classical solution of HJE is when solution is C^1

Thm u is weak KAM solution $\Leftrightarrow H(x, du(x)) = c$ is a classical solution of HJE

graph of du is an invariant set

Problem: often, we don't have classical solutions of HJE

Ex Pendulum

$$H(x, p) = \frac{1}{2} p^2 + \varepsilon \cos(2\pi x)$$

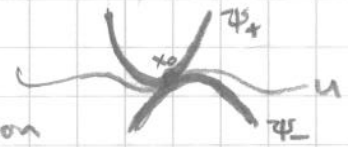
No classical solution - problems with separatrix.

Very Weak Solution of HJE $u: \mathbb{T}^N \rightarrow \mathbb{R}$ is Lipschitz $H(x, du(x)) = C$ when it exists

this is too weak - many such solutions

Def $u: \mathbb{T}^N \rightarrow \mathbb{R}$ is a viscosity solution of HJE at x_0 if for every ψ_+ (ψ_-) $\mathbb{T}^N \rightarrow \mathbb{R}$ C^1 such that $\psi_+ \geq u$ ($\psi_- \leq u$) $\psi_+(x_0) = u(x_0)$ ($\psi_-(x_0) = u(x_0)$)then $H(x_0, d\psi_+(x_0)) \leq C$ is the viscosity subsolution $H(x_0, d\psi_-(x_0)) \geq C$ is the viscosity supersolution

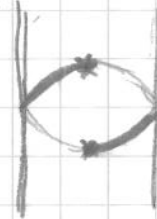
(two families of test functions)

this def must hold for each x_0 .

The classical solution is always a viscosity solution

- looking for weak KAM solution

Pendulum at separatrix
graph of du is:
backward invariant
integrates to zero



Several theorems on glides

Regularity Thm C^1 weak solution is C^1 - du is LipschitzThmif u is a classical solution of HJE, dynamics restricted to graph of du

is Lipschitz conjugate to a rotation

the u is C^2

① Intuition for near-integrable systems?

Even if you don't have a solution by KAM tori,
you have graphs of du .

② Can you obtain a family of solutions?

modify Lagrangian $L + \lambda \cdot V$ - doesn't modify dynamics,
but modifies minimizers

