3A

Introduction to KAM in the Planetary System

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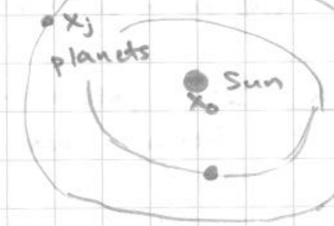


Consider $1+n$ bodies of masses $m_0, \varepsilon m_1, \dots, \varepsilon m_n$

all of the masses except m_0 will be small

position of the masses: $x_0, x_1, \dots, x_n \in \mathbb{R}^3$

Newton's equations: $\ddot{x}_j = m_0 \underbrace{\dfrac{x_j - x_0}{\|x_j - x_0\|^3}}_{\text{if } j \neq 0} + \varepsilon \sum_{k \neq j} m_k \dfrac{x_k - x_j}{\|x_k - x_j\|^3} \rightsquigarrow$ vector field of phase space in dim $6n$.

use asymptotics since $m_{\text{Jupiter}} / m_{\text{sun}} = 1/1000$

If $\underline{\varepsilon = 0}$, the system reduces to n uncoupled Kepler problems.

Ellipses around the sun.

Translational invariance $\Rightarrow$ we can take the sun to be stationary at the origin.

What happens if $0 < \varepsilon \ll 1$

For $\varepsilon = 0$, Keplerian action of the n-torus

invariants: a_j semi major axis

e_j eccentricities

For small ε, we expect that the orbits will be ellipses for a short time.
For a long time, these a_j, e_j will vary – no longer constants.

<u>Arnold's Theorem</u> (Arnold 1963, Herman, Fejoz 2004)

$\forall\, m_0, m_1, \dots, m_n \quad \forall\, a_1^0 < \dots < a_n^0, \quad \forall\, \varepsilon \ll 1$

there exists a set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions with $3n-2$ frequencies close to circular coplanar Keplerian motions with semimajor axes $a_1^0, \dots, a_n^0$

quasiperiodic motion with $3n-2$ frequencies:

$$\begin{tikzcd} t \arrow[d, mapsto] & \mathbb{R} \arrow[r, "x"] \arrow[d] & \mathbb{R}^{3(n+1)} \\ t\omega & \mathbb{T}^{3n-2} = \mathbb{R}^{3n-2}/\mathbb{Z}^{3n-2} \arrow[ur, "X"'] & \end{tikzcd}$$

$x(t) = X(t\omega)$

$\omega \in \mathbb{R}^{3n-2}$ frequency vector

these solutions are bounded: planets don't collide, aren't ejected

<u>Note</u> $|a_j(t) - a_j^0| = O(\varepsilon)$ Motion will be close to an ellipse, but not exactly.

How small does ε has to be for the proof?

Hénon – necessary condition for Arnold's prove: $\varepsilon_0 = 10^{-300}$ ← to small for theorem to be relevant to astronomy

A. Celletti & L. Chiercha – truncated version: $\varepsilon \sim 10^{-3}$

T. Caston – another careful work to get some ε.

Note: It is possible to extend this to high eccentricity orbits. Doesn't have to be near circular.

Other results – outside of this set, the motion is very not quasiperiodic.

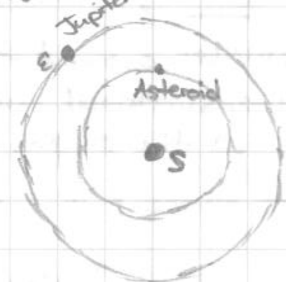
Poincaré

Laskar – 1980's / 90's – numerical studies

dynamics highly sensitive / chaotic

close passes dramatically change the later dynamics

$\underline{n=2}$ $\quad m_0 = m_2 = 1, \quad \underbrace{m_1 = 0}_{\text{restricted problem}}, \quad \varepsilon \sim \frac{1}{1000} \ll 1$



m_0, m_2 travel in ellipses.

they influence, but are not influenced by m_1.

Thm (Fejoz – Guardia – Kaloshin – Roldan)

If $0 < e_J \ll 1$, $\exists$ solution, $T > 0$

$e_A(0) < e_{min}$ (indep of e_J)

$e_A(T) > e_{max}$

an initially near-circular orbit of the asteroid which becomes more elliptical

If Jupiter has zero eccentricity, the eccentricity of the asteroid would not drift like this. Shift into a frame rotating with Jupiter's orbital period. The resulting description of the asteroid's orbit would be integrable.

We can't make e_{max} arbitrarily close to 1.

We will also need some more conditions to address resonances, e.g. $\frac{a_J}{a_A} = 3$

These resonances could give an explanation for Kirkwood gaps

places where there is unusually low density of asteroids.

If eccentricity gets large enough, the asteroid will have close encounters with Jupiter, which totally change the asteroid's orbit.

Back to Arnold's Thm:

For $\underline{\varepsilon = 0}$ $\quad$ Let $z_j = x_j - x_0$.

$$\ddot{z} = -\frac{z}{\|z\|^3}$$

Lagrange's Proof of Kepler's First Law

By symmetry, motion stays in place defined by $z(0)$ and $\dot{z}(0)$.

Write $z = x + iy$. Since it's a plane, use complex numbers.

$r = \sqrt{x^2+y^2}$

$$* \begin{cases} \ddot{x} = -x/r^3 \\ \ddot{y} = -y/r^3 \end{cases} \xrightarrow[\text{polar}]{\text{shift to}} \left. \ddot{r} = \frac{C^2 - r}{r^3} \right\}$$

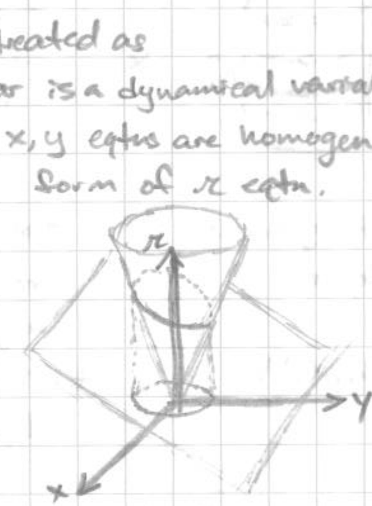
where $C = x\dot{y} - y\dot{x}$
angular momentum
constant of motion

Lagrange: if you know a solution $r(t)$,
you could also solve these* where the denominator is treated as
an explicit function of time and only the numerator is a dynamical variable.

So the solution must have the form: $\quad$ x, y eqtns are homogeneous form of r eqtn.

$$r(t) = C^2 + \alpha\, x(t) + \beta\, y(t) \qquad (\forall t) \quad \alpha, \beta \in \mathbb{R}$$

So the solution must lie on the
intersection of a $\underset{r = \sqrt{x^2+y^2}}{\underline{\text{cone}}}$ and a $\underset{r = C^2 + \alpha x + \beta y}{\underline{\text{plane}}}$.



$\therefore$ Motion must be a conic section.

Q $\quad$ Can this be extended to systems with binary stars?
Yes. Need a new small parameter - stars close, planet far
If all major axes are similar, this is a full 3-body problem.
- no perturbative solution

This system has several symmetries.
Rotation will become important.
To deal with translation & Galilean invariance, quotient out something
or - fix the sun at the origin

Lemma $\quad$ $\exists$ $\mathbb{R}$-analytic symplectic coordinates in the neighborhood of
circular horizontal (in plane) Keplerian motion $\quad (\lambda_j, \Lambda_j, x_j, y_j, x_{n+j}, y_{n+j})$
$\lambda_j \in \mathbb{T}$ is an angular, all other coordinates $\in \mathbb{R}$.

such that
$$H = \underset{\substack{\uparrow \\ \text{term that gives} \\ \text{Keplerian orbits}}}{\mathrm{Kep}(\Lambda)} + \varepsilon \sum_{j=1}^{2n} \sigma_j(\Lambda) \underbrace{\frac{x_j^2 + y_j^2}{2}}_{=:\rho_j} + \varepsilon \sum_{j,k=1}^{n} \tau_{jk}(\Lambda)\rho_j\rho_k + \varepsilon\, \mathcal{O}(\rho^3) + \mathcal{O}(\varepsilon^2)$$

Conjugate pairs:

$\lambda_j \leftrightarrow \Lambda_j$

$x_j \leftrightarrow y_j$

$x_{n+j} \leftrightarrow y_{n+j}$

if $\varepsilon = 0$, we have n uncoupled Kepler problems.

$\Lambda_j \sim \sqrt{a_j}$, λ_j tells you where along the orbit you are

x_j, y_j tell you eccentricity & direction of ellipse in the plane

x_{n+j}, y_{n+j} tell you the location of the plane in $\mathbb{R}^3$

first remarkable property of an ellipse is that it's a closed curve

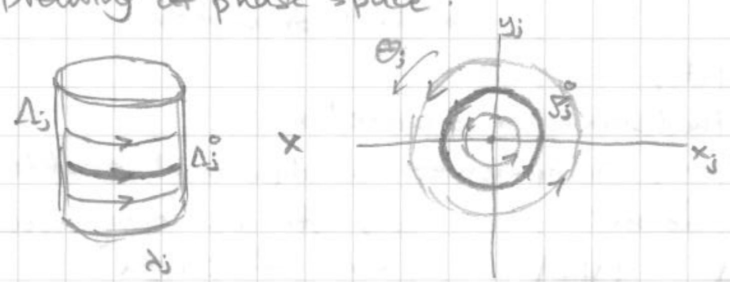
in general, this Hamiltonian system could have two frequencies $\Rightarrow$ quasiperiodic motion instead of just one

the only two potentials that have this are $\frac{1}{r}$ and r^2

all others have perihelion drift.

$$\omega = \sum_{j=1}^{n} \left(d\lambda_j \wedge d\Lambda_j + dx_j \wedge dy_j + dx_{n+j} \wedge dy_{n+j} \right)$$

Drawing of phase space:



- near the fixed point σ_j is the frequency of the precession.
- as you move away from the fixed point, the frequency is modified by the higher order terms in ρ

fast dynamics - Kepler

slow secular dynamics

radius ~ eccentricity

angle ~ direction of ellipse

$H = Kep(\Lambda) \quad + \varepsilon \sum_i \frac{1}{2} \sigma_j \rho_j$

ellipse slowly rotates

H used by Lagrange in honor of Huygen when Hamilton was 5 years old.

look in a neighborhood of a particular $\Lambda_j^\circ, \rho_j^\circ$. $\qquad \rho_j^\circ$ should be small

call $u = (\Lambda^\circ, \rho^\circ)$

If we truncated the Hamiltonian here, the system is integreble.

each u has a frequency vector

$$\alpha^\circ = \left(\frac{\partial Kep}{\partial \Lambda}, \varepsilon \left(\sigma(\Lambda) + \tau(\Lambda)\rho \right) + \mathcal{O}(\rho^2) \right) \qquad \text{note: } \frac{\partial Kep}{\partial \Lambda} = \dot{\lambda}$$

If H (including $\mathcal{O}(\varepsilon^2)$) has quasiperiodic integral curves, its frequency vector will be some α ε-close to α°

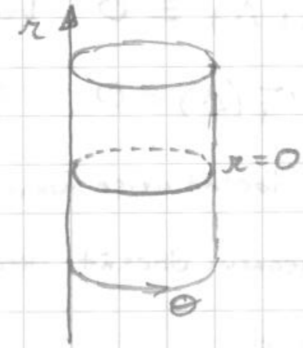
$$H_u^\circ = C_u + \alpha_u^\circ \cdot r + \mathcal{O}(r^2, \theta) \qquad \text{on } \mathbb{T}_\theta^N \times \mathbb{R}_r^{N} *$$

in a neighborhood of $r = 0$

$$\omega = \sum d\theta_j \wedge dr_j \qquad u \in \mathbb{B}_*^K, \ K = N$$

$N = 3n$

each planet has 3 frequencies:
- orbit
- precession
- rotation of plane in $\mathbb{R}^3$



Since C_u, α_u° are independent of θ,
$r = 0$ is an invariant torus with frequency vector α_u°

At $r = 0$, $\begin{cases} \dot\theta = \dfrac{\partial H_u^\circ}{\partial r} = \alpha_u^\circ \\ \dot r = -\dfrac{\partial H_u^\circ}{\partial \theta} = 0 \end{cases}$

This H_u° is the general form of an invariant torus.
Note: superscript $\circ$ means we dropped ε^2.

$$H_{\Lambda^\circ, \rho^\circ}(\lambda, \Lambda, \theta, \rho) = H(\lambda, \Lambda^\circ + \Lambda, \theta, \rho^\circ + \rho)$$

assume $\left| H_u - H_u^\circ \right| \ll 1$ in some analytic norm.
there is a more precise statement of this

<u>Thm</u> (Kolmogorov)

If $\begin{cases} u \mapsto \alpha_u^\circ \text{ is a local diffeomorphism} \\ \alpha_*^\circ \in D_{\gamma,\tau} \quad \text{Diophantine} \end{cases}$

focus on frequency for one of the u's – refer to it as $*$

then $\exists!\, u$ such that H_u has an α_*°-quasiperiodic invariant torus

$$\alpha_*^\circ \ : \ \forall k \in \mathbb{Z}^N \setminus \{0\}, \ \left| k_1 \alpha_{*,1}^\circ + \cdots + k_N \alpha_{*,N}^\circ \right| \geq \frac{\gamma}{\|k\|^\tau}$$

↳ this is the Diophantine condition ↳ more optimal conditions exist.
it prevents resonant frequencies
If $\tau > N-1$, this set is nonempty; has full measure.

<u>Idea of Proof</u>

Consider all H° of the form above and all symplectomorphisms G.
Compose. $(H^\circ, G) \mapsto H^\circ \circ G$ we want this to be H_u

$$(H_*^\circ, \text{identity}) \mapsto H_*^\circ$$

Inverse function thm. Look for $H^\circ \circ G \stackrel{?}{=} H_u$
Search for H°, G. Use Newton's method with some guess.
Linearize the equation at each step: $\displaystyle\sum_{j=1}^{N} \alpha_{*,j}^\circ \frac{\partial f}{\partial \theta_j} = g$
f from H°, G. g from H_u
Solve this PDE as a Fourier Series.

But there will be resonances.

1) $\sigma_{2n}^{0}(\Lambda) \equiv 0$. because of rotation symmetry

2) $\sum_{j=1}^{2n} \sigma_j(\Lambda) \equiv 0$ (Clairaut, Herman 90's)
2 planets

doesn't arise from any symmetry of H.

So the theorem doesn't directly apply

The way out:

Reduce the system by the symmetry of rotations.

Don't get explicit calculations for arbitrary number of planets.

You could also just look at Λ, not ρ.
It varies widely enough to pass through an α which satisfies Diophantine.