

# INTRODUCTION TO KAM IN THE PLANETARY SYSTEM

JACQUES FEJOZ



Consider  $1+n$  bodies of masses  $m_0, \epsilon m_1, \dots, \epsilon m_n$

all of the masses except  $m_0$  will be small  
position of the masses:  $x_0, x_1, \dots, x_n \in \mathbb{R}^3$

Newton's equations:  $\ddot{x}_j = m_0 \frac{x_j - x_0}{\|x_j - x_0\|^3} + \epsilon \sum_{k \neq j} m_k \frac{x_k - x_j}{\|x_k - x_j\|^3}$   $\leadsto$  vector field of phase space in dim  $6n$ .

use asymptotics since  $m_{\text{Jupiter}} / m_{\text{Sun}} = 1/1000$

If  $\epsilon = 0$ , the system reduces to  $n$  uncoupled Kepler problems.

Ellipses around the sun.

Translational invariance  $\Rightarrow$  we can take the sun to be stationary at the origin.

What happens if  $0 < \epsilon \ll 1$

For  $\epsilon = 0$ , Keplerian action of the  $n$ -torus

invariants:  $a_j$  semi major axis  
 $e_j$  eccentricities

For small  $\epsilon$ , we expect that the orbits will be ellipses for a short time.

For a long time, these  $a_j, e_j$  will vary - no longer constants.

Arnold's Theorem (Arnold 1963, Herman, Fejoz 2004)

$$\forall m_0, m_1, \dots, m_n \quad \forall a_1^0 < \dots < a_n^0, \quad \forall \epsilon \ll 1$$

there exists a set of initial conditions of positive Lebesgue measure leading to quasiperiodic motions with  $3n-2$  frequencies close to circular coplanar Keplerian motions with semimajor axes  $a_1^0, \dots, a_n^0$

quasiperiodic motion with  $3n-2$  frequencies:

$$x: \mathbb{T} \rightarrow \mathbb{R}^{3(n+1)} \quad x(t) = X(t\omega)$$

$$\downarrow \quad \downarrow \quad \nearrow X$$

$$t\omega \quad \mathbb{T}^{3n-2} = \mathbb{R}^{3n-2} / \mathbb{Z}^{3n-2} \quad \omega \in \mathbb{R}^{3n-2} \quad \text{frequency vector}$$

these solutions are bounded: planets don't collide, aren't ejected

Note  $|a_j(t) - a_j^0| = O(\epsilon)$

Motion will be close to an ellipse, but not exactly.



How small does  $\epsilon$  has to be for the proof?

Hénon - necessary condition for Arnold's prove:  $\epsilon_0 = 10^{-300}$  ← to small for theorem to be relevant to astronomy

A. Celletti & L. Chierchia - truncated version:  $\epsilon \sim 10^{-3}$

T. Castron - another careful work to get some  $\epsilon$ .

Note: It is possible to extend this to high eccentricity orbits.  
Doesn't have to be near circular.

Other results - outside of this set, the motion is very not quasiperiodic.

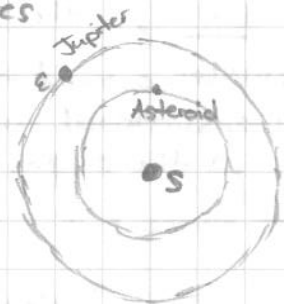
Poincaré

Las kar - 1980's/90's - numerical studies

dynamics highly sensitive/chaotic

close passes dramatically change the later dynamics

$n=2$   $m_0 = m_2 = 1$ ,  $m_1 = 0$ ,  $\epsilon \ll \frac{1}{1000} \ll 1$   
restricted problem



$m_0, m_2$  travel in ellipses.

they influence, but are not influenced by  $m_1$ .

Thm (Fejóz - Guardia - Kaloshin - Roldán)

If  $0 < \epsilon_j \ll 1$ ,  $\exists$  solution,  $T > 0$

$e_A(0) < \epsilon_{min}$  (indep of  $\epsilon_j$ )

an initially near-circular orbit of the asteroid which becomes more elliptical

$e_A(T) > \epsilon_{max}$

If Jupiter has zero eccentricity, the eccentricity of the asteroid would not drift like this. Shift into a frame rotating with Jupiter's orbital period.

The resulting description of the asteroid's orbit would be integrable.

We can't make  $\epsilon_{max}$  arbitrarily close to 1.

We will also need some more conditions to address resonances, e.g.  $\frac{a_j}{a_A} = 3$

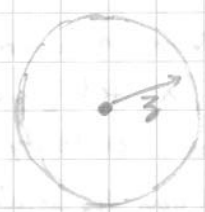
These resonances could give an explanation for Kirkwood gaps

places where there is unusually low density of asteroids.

If eccentricity gets large enough, the asteroid will have close encounters with Jupiter, which totally change the asteroid's orbit.

Back to Arnold's Theorem:

For  $\epsilon = 0$  let  $\vec{z}_j = x_j - x_0$   
 $\ddot{\vec{z}} = -\frac{\vec{z}}{\|\vec{z}\|^3}$



Lagrange's Proof of Kepler's First Law

By symmetry, motion stays in plane defined by  $\vec{z}(0)$  and  $\dot{\vec{z}}(0)$ .

Write  $\vec{z} = x + iy$ . Since it's a plane, use complex numbers.

$r = \sqrt{x^2 + y^2}$

\*  $\left\{ \begin{aligned} \ddot{x} &= -x/r^3 \\ \ddot{y} &= -y/r^3 \end{aligned} \right\}$   $\xrightarrow[\text{polar}]{\text{shift to}}$   $\ddot{z} = \frac{c^2 - r}{r^3} z$  where  $c = x\dot{y} - y\dot{x}$  angular momentum constant of motion

Lagrange: if you know a solution  $r(t)$ , you could also solve these\* where the denominator is treated as an explicit function of time and only the numerator is a dynamical variable.

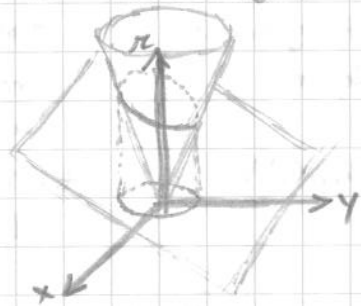
So the solution must have the form:

$r(t) = c^2 + \alpha x(t) + \beta y(t)$  ( $\forall t$ )  
 $\alpha, \beta \in \mathbb{R}$

$x, y$  eqns are homogeneous form of  $r$  eqn.

So the solution must lie on the intersection of a cone and a plane.

$r = \sqrt{x^2 + y^2}$        $r = c^2 + \alpha x + \beta y$



$\therefore$  Motion must be a conic section.

Q Can this be extended to systems with binary stars?

Yes. Need a new small parameter - stars close, planet far  
 If all major axes are similar, this is a full 3-body problem.  
 - no perturbative solution

This system has several symmetries.

Rotation will become important.

To deal with translation & Galilean invariance, quotient out something or - fix the sun at the origin

Lemma

$\exists$   $\mathbb{R}$ -analytic symplectic coordinates in the neighborhood of circular horizontal (in plane) Keplerian motion  $(\lambda_j, \Lambda_j, x_j, y_j, x_{nj}, y_{nj})$

$\lambda_j \in \mathbb{T}$  is an angular, all other coordinates  $\in \mathbb{R}$ .

such that  $H = \text{Kep}(\Lambda) + \epsilon \sum_{j=1}^{2n} \sigma_j(\Lambda) \frac{x_j^2 + y_j^2}{2} + \epsilon \sum_{i,k=1}^n \tau_{jk}(\Lambda) \rho_j \rho_k + \epsilon \mathcal{O}(\rho^3) + \mathcal{O}(\epsilon^2)$

term that gives Keplerian orbits

if  $\epsilon \neq 0$ , we have  $n$  uncoupled Kepler problems.  
 $\Delta_j \sim \sqrt{a_j}$ ,  $\lambda_j$  tells you where along the orbit you are  
 $x_j, y_j$  tell you eccentricity & direction of ellipse in the plane  
 $x_{nj}, y_{nj}$  tell you the location of the plane in  $\mathbb{R}^3$

Conjugate pairs:

$$\lambda_j \leftrightarrow \Delta_j$$

$$x_j \leftrightarrow y_j$$

$$x_{nj} \leftrightarrow y_{nj}$$

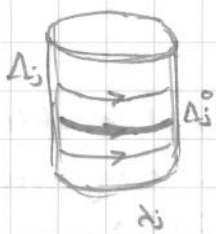
first remarkable property of an ellipse is that it's a closed curve

in general, this hamiltonian system could have two frequencies  $\rightarrow$  quasiperiodic motion instead of just one

the only two potentials that have this are  $\frac{1}{r}$  and  $r^2$   
 all others have perihelion drift.

$$\omega = \sum_{j=1}^n (d\lambda_j \wedge d\Delta_j + dx_j \wedge dy_j + dx_{nj} \wedge dy_{nj})$$

Drawing of phase space:



fast dynamics  
Kepler

$$H = K_{ep}(\Delta)$$



slow secular dynamics  
radius  $\sim$  eccentricity  
angle  $\sim$  direction of ellipse  
 $+\epsilon \sum \frac{1}{2} \sigma_j p_j$   
 ellipse slowly rotates

- near the fixed point  $\sigma_j$  is the frequency of the precession.
- as you move away from the fixed point, the frequency is modified by the higher order terms in  $p$

[H used by Lagrange in honor of Huygen when Hamilton was 5 years old.]

look in a neighborhood of a particular  $\Delta_j^0, p_j^0$ .  
 call  $u = (\Delta^0, p^0)$

$p_j^0$  should be small

If we truncated the hamiltonian here, the system is integrable.  
 each  $u$  has a frequency vector

$$\alpha^0 = \left( \frac{\partial K_{ep}}{\partial \Delta}, \epsilon (\sigma(\Delta) + \tau(\Delta)p) + \mathcal{O}(p^2) \right) \quad \text{note: } \frac{\partial K_{ep}}{\partial \Delta} = \dot{\lambda}$$

If  $H$  (including  $\mathcal{O}(\epsilon^2)$ ) has quasiperiodic integral curves, its frequency vector will be some  $\alpha$   $\epsilon$ -close to  $\alpha^0$

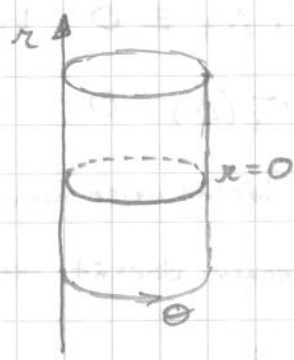
$$H_u^0 = C_u + \alpha_u^0 \cdot \pi + \mathcal{O}(\pi^2, \theta)$$

on  $\mathbb{T}_\theta^N \times \mathbb{R}_\pi^{N*}$   
in a neighborhood of  $\pi = 0$   $N = 3n$

$$\omega = \sum d\theta_j \wedge d\pi_j \quad u \in \mathbb{B}_*^K, K=N$$

Since  $C_u, \alpha_u^0$  are independent of  $\theta$ ,  
 $\pi=0$  is an invariant torus with  
frequency vector  $\alpha_u^0$

$$\text{At } \pi=0, \begin{cases} \dot{\theta} = \frac{\partial H_u^0}{\partial \pi} = \alpha_u^0 \\ \dot{\pi} = -\frac{\partial H_u^0}{\partial \theta} = 0 \end{cases}$$



- each planet has 3 frequencies:
- orbit
- precession
- rotation of plane in  $\mathbb{R}^3$

This  $H_u^0$  is the general form of an invariant torus.  
Note: superscript 0 means we dropped  $\epsilon^2$ .

$$H_{\Lambda^0, p^0}(\lambda, \Lambda, \theta, p) = H(\lambda, \Lambda^0 + \Lambda, \theta, p^0 + p)$$

assume  $|H_u - H_u^0| \ll 1$  in some analytic norm.  
there is a more precise statement of this

### Thm (Kolmogorov)

If  $\{ u \mapsto \alpha_u^0 \}$  is a local diffeomorphism  
 $\alpha_{*}^0 \in D_{\pi, \tau}$  Diophantine

then  $\exists! u$  such that  $H_u$  has an  $\alpha_{*}^0$ -quasiperiodic invariant torus

$$\alpha_{*}^0 : \forall k \in \mathbb{Z}^N \setminus \{0\}, |k_1 \alpha_{*,1}^0 + \dots + k_N \alpha_{*,N}^0| \geq \frac{\tau}{\|k\|^\tau}$$

$\hookrightarrow$  this is the Diophantine condition  
it prevents resonant frequencies  $\hookrightarrow$  more optimal conditions exist.  
If  $\tau > N-1$ , this set is nonempty; has full measure.

### Idea of Proof

Consider all  $H^0$  of the form above and all symplectomorphisms  $G$ .  
Compose.  $(H^0, G) \mapsto H^0 \circ G$  we want this to be  $H_u$

$$(H_{*}^0, \text{identity}) \mapsto H_{*}$$

Inverse function thm. Look for  $H^0 \circ G \stackrel{?}{=} H_u$

Search for  $H^0, G$ . Use Newton's method with some guess.

Linearize the equation at each step:  $\sum_{j=1}^N \alpha_{*,j}^0 \frac{\partial b}{\partial \theta_j} = g$   
 $b$  from  $H^0, G$ .  $g$  from  $H_u$

Solve this PDE as a Fourier Series.

But there will be resonances.

1)  $\sigma_{2n}^0(\Delta) \equiv 0$  because of rotation symmetry

2)  $\sum_{j=1}^{2n} \sigma_j(\Delta) \equiv 0$  (Clairaut, Herman 90's)  
Zplanets

doesn't arise from any symmetry of  $H$ .

So the theorem doesn't directly apply

The way out:

Reduce the system by the symmetry of rotations.

Don't get explicit calculations for arbitrary number of planets.

You could also just look at  $\Delta$ , not  $p$ .

It varies widely enough to pass through an  $\alpha$  which satisfies Diophantine.