5A

INTRODUCTION TO KAM APPLIED TO PDE

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References

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- S. Kuksin: Nearly Integrable Inf Dimensional Hamiltonian Systems: Springer LNM 1556 (1991)
- J Pöschel: "A KAM-Theorem for some Nonlinear PDE's" Ann. Sc. Norm Pisa 23 (1996)
- C.E. Wayne: "An Intro KAM theory": Lec. App. Math, vol 31: AMS (1996)

proofs are long - we won't be able to do an entire proof here

there isn't one KAM theory for all relevant circumstances

you have to find the one relevant to your problem

Def Hamiltonian PDE

Suppose $\mathcal{H}$ is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and J - anti-symmetric invertible operator.

A Hamiltonian PDE has the form

$$\partial_t u = J\nabla H$$

∇H is defined by $\frac{d}{d\varepsilon} H(u+\varepsilon h)\big|_{\varepsilon=0} = \langle \nabla H, h\rangle$

We will focus on a simple example: nonlinear Schrödinger (NLS)

$$i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + V(x)\,u(x) + |u|^{2r}u$$

$$u(0,t) = u(\pi,t) = 0$$

write the eq. as separate real & imaginary parts $u(x,t) = q(x,t) + i\,p(x,t)$

$$i\frac{\partial q}{\partial t} - \frac{\partial p}{\partial t} = \frac{\partial^2 q}{\partial x^2} + i\frac{\partial^2 p}{\partial x^2} + V(x)q + iV(x)p + (p^2+q^2)^r(q+ip)$$

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p\end{pmatrix} = \begin{pmatrix} \frac{\partial^2 p}{\partial x^2} + V(x)p(x) + (p^2+q^2)^r p \\ -\frac{\partial^2 q}{\partial x^2} - V(x)q(x) - (p^2+q^2)^r q \end{pmatrix}$$

work in Sobolev spaces. (have to be careful about domains)

$$\mathcal{H} = \mathcal{H}_0^1(0,\pi)\times\mathcal{H}_0^1(0,\pi)$$

superscript - this many derivatives in L2

subscript - boundary conditions are zero.

$$\left\langle \begin{pmatrix} q\\ p\end{pmatrix}, \begin{pmatrix}\tilde q\\ \tilde p\end{pmatrix}\right\rangle = \int_0^\pi \left(p(x)\tilde p(x) + q(x)\tilde q(x)\right)dx$$

$$H(p,q) = \frac{1}{2}\int_0^\pi \left(p_x^2 + q_x^2 - V(x)\left(p^2(x)+q^2(x)\right)\right)dx - \frac{1}{2(r+2)}\int_0^\pi (p^2+q^2)^{r+1}\,dx$$

compute the gradient:

$$\frac{d}{d\varepsilon} H(p+\varepsilon h, q+\varepsilon g)\Big|_{\varepsilon=0} = \int_0^\pi \Big(\underbrace{p_x h_x + q_x g_x}_{\substack{\text{integrate by parts}\\ -p_{xx}h - q_{xx}g}} - (ph-qg)V(x)\Big) - \int_0^\pi (p^2+q^2)^r(ph-qg)\,dx$$

$$\Rightarrow \nabla H = \begin{pmatrix} -q_{xx} - qV - (p^2+q^2)^r q \\ -p_{xx} - pV - (p^2+q^2)^r p \end{pmatrix}$$

Take $J\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}$

Note: We don't need $J^2 = $ Identity

We should have: $\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = J \nabla H$.

$$J\nabla H = \begin{pmatrix} p_{xx} + pV + (p^2+q^2)^r p \\ -q_{xx} - qV - (p^2+q^2)^r q \end{pmatrix}$$

So the PDE is a Hamiltonian field theory.

This is not the only way to represent NLS in the Hamiltonian form.

<u>Remark</u>

Take a Hilbert space of complex valued functions: $\tilde{\mathcal{H}} = \mathcal{H}_0^1([0,1] : \mathbb{C})$

$$\langle\langle u, w \rangle\rangle = \mathrm{Re} \int_0^\pi u(x)\overline{w}(x)\,dx \qquad \tilde{J} = -i.$$

This also produces NLS as a Hamiltonian system.

<u>Other Examples</u>

① Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(u) \text{ ; } g \text{ analytic, } g(0) = 0 \text{ ; } u(0) = u(\pi) = 0$$

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{\partial^2 u}{\partial x^2} + g(u) \end{pmatrix}$$

This is Hamiltonian with:

$$H(u,v) = \frac{1}{2}\int_0^\pi \left(\frac{1}{2}\left(v(x)^2 + (\partial_x u)^2\right) + G(u)\right) dx$$

G - anti-derivative of g

② KdV equation

shows us why we need to generalize our symplectic operator J

$$H_{KdV} = \int \frac{1}{2} u_x^2 - \frac{1}{6} u^3 \qquad J = \partial_x$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0$$

Q What if domain is not compact or higher dimensional?

much harder to do KAM

Q Do people use Banach spaces?

no for KAM - instead use a scale of Hilbert spaces

Canonical Transformation

change coordinates while keeping the Hamiltonian structure

we have: $\mathcal{H}$, $\langle\cdot,\cdot\rangle$, J $\Rightarrow$ build 2-form $\omega = -\langle J^{-1}du, du\rangle$

Suppose there's another ← often, $J = J^{-1}$

$$\tilde{\mathcal{H}}, \langle\langle\cdot,\cdot\rangle\rangle, \tilde{J} \Rightarrow \tilde{\omega}$$

A map between these $\phi: \mathcal{H} \to \tilde{\mathcal{H}}$ is a canonical transformation if the pullback $\phi^*\tilde{\omega} = \omega$

Ex Harmonic Oscillator

$$\mathcal{H} = \mathbb{R}^2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \left\langle \begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} \right\rangle = q\tilde{q} + p\tilde{p}$$

$$\tilde{\mathcal{H}} = \mathbb{C}, \quad \tilde{J} = -i, \quad \langle\langle z, w\rangle\rangle = \mathrm{Re}(z\bar{w})$$

$$\phi: \begin{pmatrix} q \\ p \end{pmatrix} = q + ip = z$$

$$\begin{aligned}
(\phi^*\omega)\left[\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}\right] &= -\langle\langle i(q+ip), (\tilde{q}+i\tilde{p})\rangle\rangle \\
&= -\langle\langle (-p+iq), (\tilde{q}+i\tilde{p})\rangle\rangle \\
&= -(-p\tilde{q} + q\tilde{p}) = p\tilde{q} - q\tilde{p} \\
&= -\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} \right\rangle = \omega\left[\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}\right]
\end{aligned}$$

In the new coordinates, the equations will also be Hamiltonian.

$$H(q,p) = \tfrac{1}{2}\Omega(p^2+q^2) \rightsquigarrow \dot{q} = p, \; \dot{p} = -q$$

$$\tilde{H}(z) = \tfrac{1}{2}\Omega\langle\langle z,z\rangle\rangle = \tfrac{1}{2}\Omega|z|^2 \rightsquigarrow \dot{z} = -i\Omega z$$

Alternatively, $\psi\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \varphi \\ I \end{pmatrix} = \begin{pmatrix} \tan^{-1}(p/q) \\ \frac{1}{2}(p^2+q^2) \end{pmatrix}$

Show this is a canonical transformation

$\psi: \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}_+$ $\qquad \tilde{\tilde{\omega}} = d\phi \wedge dI$

In terms of (I, φ), $\quad H(I,\varphi) = \Omega I$

$$\dot{\varphi} = \frac{\partial H}{\partial I} = \Omega, \quad \dot{I} = -\frac{\partial H}{\partial \varphi} = 0$$

These are action-angle variables. The actions (I) are conserved.

Theorem (Poincaré)

Let ϕ_H^t be the flow associated to a Hamiltonian system. $\dot{u} = J\nabla H$

For any t, ϕ_H^t is a canonical transformation.

KAM is perturbative – we will be interested in cases where the flow is small.

How do we compute

$$f\circ\Phi_{\varepsilon F}^{t=1}(u) = \overbrace{f(u)}^{\text{identity}} + \Big(\overbrace{f\circ\Phi_{\varepsilon F}^{t=1}(u) - f(u)}^{\text{correction. Use fundamental thm of calculus.}}\Big)$$

$$= f(u) + \int_0^1 \frac{d}{dt}\left(f\circ\Phi_{\varepsilon F}^{t}\right)dt$$

this is really the Poisson bracket

$$= f(u) + \varepsilon\int_0^1 \left(Df\circ\Phi_{\varepsilon F}^{t}\right)\cdot J\nabla F\circ\Phi_{\varepsilon F}^{t}\,dt$$

This is still evaluated at new coordinates. Repeat the process.

$$= f(u) + \varepsilon\underbrace{Df(u)\cdot J\nabla F(u)}_{\{f,F\}(u)}$$
$$+\varepsilon^2\int_0^1\int_0^t \{\{f,F\},F\}\circ\Phi_{\varepsilon F}^{s}\,ds\,dt$$

Integrable Systems

Remark Hamiltonian is constant along orbits - constant of motion.

Remark For an integrable system with bounded orbits, there exists a canonical transformation such that in terms of the new variables (I,ϕ), the Hamiltonian depends only on the actions.

$$H(I,\phi) = h(I)$$

$$\dot{I} = -\frac{\partial H}{\partial\phi} = 0 \Rightarrow I(t) = I(0)$$

$$\dot{\phi} = \frac{\partial H}{\partial I}(I(0)) \Rightarrow \phi(t) = \Omega(I_0)t + \phi_0$$

where $\Omega(I) = \frac{\partial h}{\partial I}$

All solutions are periodic or quasiperiodic.

Suppose we have a Hamiltonian that is a perturbation away from an integrable system,

$$H(I,\phi) = h(I) + \varepsilon f(I,\phi)$$

Question Do the invariant tori of the integrable system survive when $\varepsilon\neq 0$?

(Finite Dimensional)

KAM Theorem If $H = h(I) + \varepsilon f(I,\phi)$ is defined on $(I,\phi)\in U\times\mathbb{T}^N$ for $U\subset\mathbb{R}^N$ and if for some (γ,τ): $|\Omega(I_0)\cdot n| \geq \frac{\gamma}{|n|^\tau}$ for all $n\in\mathbb{Z}^N\setminus\{0\}$

and if $\frac{\partial^2 h}{\partial I\partial I}$ is invertible

then for ε sufficiently small, the perturbed system has an invariant torus with frequency $\Omega(I_0)$

$|\varepsilon| < \varepsilon_0(\gamma,\tau,N)$ quasiperiodic orbit

Challenge: Diophantine frequencies exist if $\tau > N-1$. What if $N=\infty$?

We would like to prove this sort of theorem for PDEs.

But $\varepsilon_0(N) \xrightarrow[N\to\infty]{} 0$

How can we get around this - at least for simple examples?

<u>Ex.</u> Nonlinear Schrödinger

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + V(x)u + \varepsilon |u|^{2p} u \qquad u = u(x,t), \quad u(0,t) = u(\pi,t) = 0$$

Treat the nonlinear term as the perturbation. Linear equation is integrable.

<u>$\varepsilon = 0$</u> $\{\psi_j\}$, $\{\Omega_j\}$ eigenfunctions & eigenvalues

$L\psi = -\psi'' + V(x)\psi \; ; \; \psi(0) = \psi(\pi) = 0.$

$u_j^P(x,t) = e^{-i\Omega_j t}\psi_j(x)$ periodic for each j

$u^{QP}(x,t) = \sum_{j=1}^{N} a_j e^{-i\Omega_j t}\psi_j(x)$ quasi-periodic

Only perturb solutions with finitely many frequencies.

Even though solutions with infinitely many frequencies exist

<u>Remark</u> We know a lot about these eigenvalues.

$\Omega_j = j^2 + \bar{V} + \eta_j \qquad \bar{V} = \int_0^\pi V(x)\,dx$

If $V \in H^s(0,\pi)$, $\{\eta_j\} \in h^s = \{\eta \mid \sum_{j=1}^{\infty} \eta_j^2 j^{2s} < \infty\}$

By changing V, we can pick finitely many arbitrary eigenvalues

This takes the role of the invertible hessian.

$H = \frac{1}{2}\int_0^\pi \left(|u_x|^2 + V(x)|u|^2\right) du$

Change variables: $u(x,t) = \sum_{j=1}^{\infty} z_j(t)\psi_j(x) \quad z_j \in \mathbb{C}$

(canonical) $\quad H(z,\bar{z}) = \frac{1}{2}\sum_{j=1}^{\infty} \Omega_j |z_j|^2 \qquad J = -i$

$\dot{z}_j = -i\Omega_j z_j \qquad j = 1, \dots, \infty$ infinitely many harmonic oscillators

these are great if z_j is small. Otherwise, use action angle variables.

For the first D components, make a further canonical transformation.

$I_j = \frac{1}{2}|z_j|^2 \; ; \; \varphi_j = \arg(z_j)$

Integrable (unperturbed) system:

$$H(I,\varphi,z,\bar{z}) = \sum_{j=1}^{D} \Omega_j I_j + \frac{1}{2}\sum_{j=D+1}^{\infty} \Omega_j |z_j|^2$$

Perturbed Hamiltonian also has: $\frac{\varepsilon}{4(p+1)}\int_0^\pi |u|^{2(p+1)}\,dx$

write it as $+\varepsilon F(I,\varphi,z,\bar{z})$

For $\varepsilon = 0$: Equations of Motion

$$\left.\begin{aligned}\dot{\varphi}_j &= \Omega_j \\ \dot{I}_j &= 0\end{aligned}\right\} j = 1, \dots, D \qquad \dot{z}_j = -i\Omega_j z_j \quad j = D+1, \dots, \infty$$

this has a D-dim invariant torus $I(t) = I_0$, $\varphi(t) = \Omega t + \varphi_0$, $z(t) = 0$

↑ only first D of the Ω_j

It's more convenient to work close to the origin. $I \to I_0 + \tilde{I}$

$$H(\tilde{I}, \varphi, z, \bar{z}) = c_0 + \sum_{j=1}^{D} \Omega_j \tilde{I}_j + \frac{1}{2}\sum_{j=D+1}^{\infty} \Omega_j |z_j|^2 + \varepsilon \tilde{F}(\tilde{I}, \varphi, z, \bar{z})$$

Stop writing the ~'s and c_0.

Study this Hamiltonian system near $I=0$, $z=0$.

Goal: Show that (H_ε) has a D-dim invariant torus near $I=0, z=0$.

Method: Find a canonical transformation: $\Phi_{\varepsilon\chi}$ such that

$$\tilde{H} = (H \circ \Phi_{\varepsilon\chi})(I, \varphi, z, \bar{z}) = \sum_{j=1}^{D} \Omega_j I_j + \frac{1}{2}\sum_{j=D+1}^{\infty} \Omega_j |z_j|^2 + \varepsilon F^{>}(I, \varphi, z, \bar{z})$$

The equations of motion for $\tilde{H}$:

$$\dot{I} = -\frac{\partial \tilde{H}}{\partial \varphi} = -\varepsilon \frac{\partial F^{>}}{\partial \varphi} \qquad \dot{z}_j = -i\frac{\partial \tilde{H}}{\partial \bar{z}_j} = i\Omega_j z_j - i\varepsilon \frac{\partial F^{>}}{\partial \bar{z}_j}$$

$$\dot{\varphi} = \frac{\partial \tilde{H}}{\partial I} = \Omega + \varepsilon \frac{\partial F^{>}}{\partial I}$$

Last terms on the right of each equation vanish if $F^{>}$ is quadratic in z, I.

since we're looking near $I = z = 0$.

We only have to elliminate low order terms in I, z.

Expand: $$F(I, \varphi, z, \bar{z}) = \underbrace{\sum_{\substack{n \in \mathbb{Z} \\ 2|m| + |\mu + \bar{\mu}| \le 2}} \hat{F}(m, n, \mu, \bar{\mu})\, e^{in\cdot\varphi} I^m z^\mu \bar{z}^{\bar{\mu}}}_{\text{up to quadratic terms}} + \underbrace{F^{>}(I, \varphi, z, \bar{z})}_{\text{higher terms}}$$

Fourier in φ

Taylor in I, z

We can compute the effects of a canonical transformation to lowest order from a small Hamiltonian using the Poisson bracket. (recall from last time)

$$H \circ \Phi_{\varepsilon\chi}(I, \varphi, z, \bar{z}) = H(I, \varphi, z, \bar{z}) + \varepsilon\{\chi, H\} + \underbrace{\mathcal{O}(\varepsilon^2)}_{\text{forget}}$$

$$= \sum_{j=1}^{D} \Omega_j |z_j|^2 + \underbrace{\varepsilon \sum_{\substack{n \in \mathbb{Z} \\ 2|m| + |\mu + \bar{\mu}| \le 2}} \cdots}_{} + \varepsilon F^{>}(\cdots) + \underbrace{\varepsilon\{\chi, H\}}_{} + \mathcal{O}(\varepsilon^2)$$

Choose χ so these cancel.

Expand: $$\chi(I, \varphi, z, \bar{z}) = \sum_{\substack{n \in \mathbb{Z} \\ 2|m| + |\mu + \bar{\mu}| \le 2}} \hat{\chi}(m, n, \mu, \bar{\mu})\, e^{in\cdot\varphi} I^m z^\mu \bar{z}^{\bar{\mu}}$$

Compute: $$\{\chi, H\} = \sum_{\cdots} -i\Big(\underset{\uparrow \le D}{n \cdot \Omega} + \underset{\uparrow > D}{(\mu - \bar{\mu}) \cdot \Omega}\Big)\, \hat{\chi}(m, n, \mu, \bar{\mu})\, e^{in\cdot\varphi} I^m z^\mu \bar{z}^{\bar{\mu}}$$

This means we should choose χ:

$$\chi(I, \varphi, z, \bar{z}) = -i \sum_{\substack{n \in \mathbb{Z} \\ 2|m| + |\mu + \bar{\mu}| \le 2}} \frac{\hat{F}(m, n, \mu, \bar{\mu})\, e^{in\cdot\varphi} I^m z^\mu \bar{z}^{\bar{\mu}}}{\Big(\underset{\uparrow \le D}{n \cdot \Omega} + \underset{\uparrow > D}{(\mu - \bar{\mu}) \cdot \Omega}\Big)}$$

If $n=0$, $\mu=\bar{\mu}$, this is an additional integrable piece – exclude this

We need some sort of Diophantine condition.

What allows us to do this is that we have greatly restricted the $m, \mu, \bar{\mu}$

Melnikov Condition (M)

$$\left| n \cdot \Omega_{\le D} + (\mu-\bar{\mu}) \cdot \Omega_{>D} \right| \ge \frac{\gamma \langle \mu-\bar{\mu} \rangle_2}{(1+|n|)^{\tau}} \quad \text{for all } n \in \mathbb{Z}^D \setminus \{0\}, \ |\mu+\bar{\mu}| \le 2$$

$$\text{where } \langle \mu-\bar{\mu} \rangle_2 = \left| \sum_{j=D+1}^{\infty} (\mu-\bar{\mu})_j \, j^2 \right| + 1$$

Most dangerous case is when both μ and $\bar{\mu}$ have length one,

$\mu = \delta_k$, $\bar{\mu} = \delta_l$

$\Rightarrow (\mu-\bar{\mu}) \cdot \Omega_{>D} = \Omega_k - \Omega_l \simeq k^2 - l^2 \gtrsim \langle \mu - \bar{\mu} \rangle_2$

Two Key Facts

① Condition (M) ensures that the sum for χ converges.

② There are "lots" of frequencies that satisfy (M).

Why is ② true?

Consider all $\Omega \in B_1 \subset \mathbb{R}^D$

Claim If $\tau > D+3$ and γ is sufficiently small,

then the measure of the set of Ω that violates (M) is $\mathcal{O}(\gamma)$

Estimate $B(n,\mu,\bar{\mu})$ = measure of set violating (M) for fixed $n, \mu, \bar{\mu}$

└ bad set

$$\text{meas}\left(B(n,\mu,\bar{\mu})\right) \le \frac{C_D \, \gamma \langle \mu-\bar{\mu} \rangle_2}{(1+|n|)^{\tau}}$$

Sum over $n \ne 0$, μ satisfying $|\mu+\bar{\mu}| \le 2$.

Lemma There exists $\Theta > 0$ such that if (M) is violated,

$$|n| \ge \Theta \left| \mu - \bar{\mu} \cdot \Omega_{>} \right|$$

i.e. $n \cdot \Omega_{\le} + (\mu-\bar{\mu}) \cdot \Omega_{>}$ is small, the two terms must be opposite sign and similar magnitude.

Focus on dangerous case $\mu = \delta_k$, $\bar{\mu} = \delta_l$.

$$\text{meas} = \sum_{n \ne 0} \sum_{k,l} \frac{C_D \, \gamma \, \langle \mu-\bar{\mu} \rangle_2}{(1+|n|)^{\tau}} \le \tilde{C}_D \sum_{n \ne 0} \frac{\gamma \, n^4}{(1+|n|)^{\tau}} < C\gamma$$

(for $\tau = 3$?

this sum is finite – only N^2 terms

Remark

this depends on how the linear frequencies grow with j – should be superlinear

coincident normal frequencies (periodic boundary conditions) or

almost coincident normal frequencies (higher dimensions) also cause problems.

→ need more KAM theorems.

③ What if you have more complicated integrable behavior?

see KAM for KdV - book by Kappeler, Pöschel

③ What if there's a continuous spectrum?

unless miraculous cancellation, no KAM theorem

some multiple will couple into continuum → decay to radiation

③ Infinite-Dimensional Tori?

very few results

need something much stronger than (M)

results use a nonlinear term that smooths.