6A

INTRODUCTION TO ARNOLD DIFFUSION

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What is diffusion?
  how small forces produce large effect after a long time
  large changes in energy or action
  if you apply a periodic perturbation to the system, will it average out or accumulate?
Have Hamiltonian system that depends on time.
  if autonomous, Hamiltonian conserved → apply results on an energy surface
write system so unperturbed is in action-angle variables
$\varepsilon = 0$: invariant tori - quasiperiodic in ϕ's
KAM: many tori remain after perturbation
we will focus on ones that don't remain
Poincaré Map
  take snapshot of the dynamics at each period of the Hamiltonian
see if motion is changed to $O(1)$ due to a perturbation which is $O(\varepsilon)$
  time it takes for this to happen = Arnold diffusion time
60's theory: perturbations average out for
  most initial conditions (KAM) or long time
  Arnold gives an example of different behavior
can we tell when accumulation will happen? can we design for/against this?
Role of dimension:
  $d=1$ - ex. of swing
    push at resonant frequency → amplitude grows → different resonant frequency
    finite amplitude motion
    KAM tori are preserved → diffusion is not possible
    there are gaps where chaos lives; size of gap $O(\varepsilon^{1/2})$
  2 or more degrees of freedom
    even though invariant tori still exist, you can move around them
    new orbits that move $O(1)$ for an $O(\varepsilon)$ perturbation
What do we do?
  (1) find geometric landmarks
  (2) local analysis near them
  (3) look for connections between them - homoclinic/heteroclinic orbits
  (4) approximate orbits that shadow real orbits
Arnold's Example:
  $d=2$: 2 actions, 2 angles, time   2 parameters: ε, μ.
  $$H = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon(\cos\phi_1 - 1) + \varepsilon\mu(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$$

  Claim: $\exists C$, $\forall \varepsilon > 0$ sufficiently small, $\exists$ an orbit $(I(t), \phi(t))$, $\exists$ time T
    such that $|I_2(T) - I_2(0)| > C$ Finite motion from infinitesimal

$\varepsilon = 0$: integrable, I_1 & I_2 conserved

$\varepsilon > 0$, $\mu = 0$: degrees of freedom separable - pendulum & harmonic oscillator

Pick a 1-dim invariant torus of I_2, the separatrix of I_1 means that the invariant torus has a homoclinic orbit stable & unstable manifolds coincide.

$\varepsilon > 0$, $\mu \neq 0$ small:

doesn't affect invariant torus of I_2.

stable & unstable manifolds are changed

they now cross with each other and with the invariant manifolds of neighboring tori

make a chain of unstable manifold of torus 0 → stable manifold of torus 1
→ unstable manifold of torus 1 → stable manifold of torus 2
→ unstable manifold of torus 2 → stable manifold of torus 3

this travels a long distance in I_2.

shadowing theorem to show that there is an orbit that does this

Q how general is this result?

Remark second parameter μ must be chosen after ε.

$\mu = O(e^{-c/\varepsilon})$ exponentially smaller than μ
↳ so neighboring manifolds cross

Open problem if there is only one parameter

could take $\mu = 1$ or $\varepsilon = 1$ or ...

$\varepsilon = 1$: Big Gaps Problem

regions of chaos between tori are larger than distance moved by the heteroclinic tangle

use geometric methods

tools
- transition whiskered tori
- topological windows
- local variation methods - and global
- normally hyperbolic invariant manifolds
- etc.

Sample models:

Mather model - two time scales

gaps are small, so Arnold's method works

Restricted elliptic 3-body problem

no gaps

Big Gaps

more general - most exs need some numerics

Mather Problem

perturbed geodesic flow

energy preserved - even unperturbed stable & unstable manifolds cross with neighbors

add a time-periodic potential

for generic V, there are orbits whose energy $\to \infty$

Big Gaps Problem

pendulums + rotators + small periodic coupling

general - combination of arbitrary number of 2 kinds of things, any coupling

assume:

- functions uniformly C^1
- twist condition $\partial^2 h_0 / \partial I^2$ (NO LONGER NEEDED)
- homoclinic orbit at $(0,0)$

prove result: $\forall \varepsilon > 0$, $O(1)$ change in orbit

Normally Invariant Hyperbolic Manifold (NIHM)

invariant manifold

dynamics near manifold is dominated by expansion/contraction normal to the manifold

these will be preserved by small perturbations

will $\varepsilon = 0$, stable & unstable manifolds coincide - homoclinic

collection of all invariant tori together form a NIHM - inside, motion is quasiperiodic

for $\varepsilon > 0$, these manifold still exist

study dynamics inside manifold - almost integrable

if this function L has no critical points, stable & unstable manifolds will intersect transversally

Scattering Map

when in NIHM, each point in intersection has one heteroclinic connection to another point in NIHM

use this to make a map - shows intersections of invariant manifolds

$\varepsilon = 0$: scattering maps invariant tori to itself

only homoclinic connections - no heteroclinic connections

$\varepsilon > 0$: find a tori that has heteroclinic connections

map is symplectic, etc. $\to$ explicit formulas for it

so you can explicitly find heteroclinic connections

Inner Dynamics

what happens inside NIHM?

consider Hamiltonian constrained inside NIHM

first do a bunch of averages, then apply KAM

far from resonance - KAM

at resonance - do KAM inside each island chain

Compute image of one of these new tori under scattering map - intersects others

Shadowing then to get an orbit that follows a chain.

Goal: Can we get a similar result without as much knowledge of inner dynamics?
Don't bother showing inner tori exist.
Just use shadowing lemma.

Make a pseudo orbit:
map in inner dynamics, then scattering map, then inner, then scattering, etc.
if one of these moves $O(1)$, shadowing lemma guarantees an actual orbit also moves $O(1)$

Make an orbit of the scattering map
repeatedly apply scattering map only - explicit formula
if the inner dynamics are recurrent, we can use shadowing lemma

Formula for a scattering map looks like Euler's method for a vector field - can solve using the vector field instead.

We have Arnold diffusion if:
NHIM Λ_ε
homoclinic channel Γ_ε and scattering map σ_ε
action angle coords in Λ_ε

No conditions on inner dynamics - just Hamiltonian.
Either tori conserved → earlier thm works
or not conserved → diffusion in inner dynamics too

Q Do you get $\langle x^2 \rangle \sim t$?
the methods show that long orbits exist, not how common they are.
How long does it take this method to happen? $O(1/\varepsilon)$ in this case

Q Scattering map has a series in ε. What if ε isn't too small?
Better to compute scattering maps } → stable & unstable manifolds crossings
directly for your value of ε.

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Same problem as before – but different approach – Variational Methods
deep connections to weak KAM theory

Arnold Diffusion

$H_\varepsilon(x,p,t) = H_0(p) + \varepsilon H_1(x,p,t) \qquad x \in \mathbb{T}^d,\ p \in \mathbb{R}^d,\ t \in \mathbb{T}$

We often look at the time-one map ϕ^1_H – Poincaré section at each period of t.

(Hypothesis) For typical H_0 and εH_1, there is topological instability in p
$d \geq 2$ no topological barriers

this is sort of an anti-KAM theorem
KAM – most initial conditions' behavior remains on an invariant torus
this says that the system is dramatically changed anyway

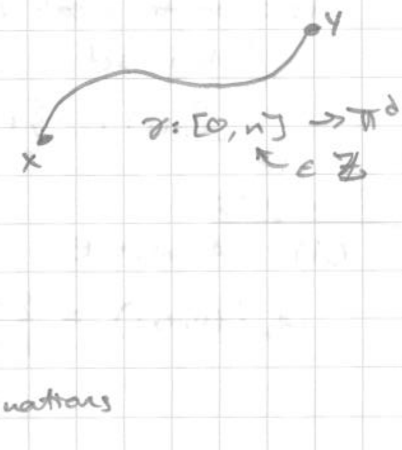
Conditions on hypothesis – like conditions of KAM.

- $\partial_p^2 H_0$ nonsingular – so we have a lagrangian
  restrict to $\partial_p^2 H_0$ positive definite – this includes classical methods
- regularity C^ω – hard – mostly open problems
  C^r – reachable theorems
- H – Tonelli: convex in p, superlinear, generates a complete flow
- Consider the Lagrangian
  $\mathcal{L}(x,v,t) = \sup \{ p \cdot v - H(x,p,t) \}$

① Understand "elementary invariant sets"
② Understand the relations between them
i.e. homoclinic & heteroclinic orbits

Poincaré's approach to dynamical systems

Look for curves that minimize the lagrangian action
with fixed endpoints & time



Curves are in configuration space – $\mathbb{T}^d$

Define $h^n(x,y) = \inf\limits_{\substack{\gamma(0)=x \\ \gamma(t)=y}} \int_0^n \mathcal{L}(\gamma,\dot\gamma,t)\,dt$

Tonelli Thm minimizers exist
Call it γ. γ is C^2, solvers Euler – lagrange Equations

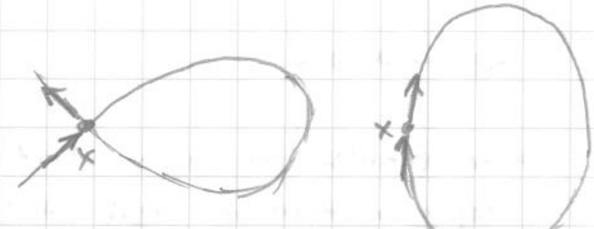
Connection to Hamiltonian mechanics:
If $h^n(x,\cdot)$ and $h^n(\cdot,y)$ are differentiable at y and x respectively

$\partial_1 h^n(x,y) = -\partial_v \mathcal{L}(\gamma(0),\dot\gamma(0),0)\,,\quad \partial_2 h^n(x,y) = \underbrace{\partial_v \mathcal{L}(\gamma(n),\dot\gamma(n),n)}_{\text{momentum } p}$

Periodic orbits

must be a minimizer of $h^n(x,x)$.

$h^n(x,x)$ is not a periodic orbit since it may not return with the same velocity



$$x_0 \in \operatorname*{argmin}_{x \in \mathbb{T}^d} h^n(x,x) \Rightarrow (x_0, \partial_2 h^n(x_0,x_0)) \text{ is periodic under } \Phi_H^1$$

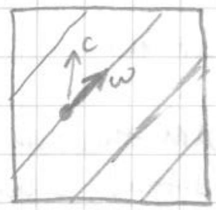
Problem - this often gives the trivial periodic orbit that does not move

Penalize not going away from the initial position.

$\mathcal{L}_c(x,v,t) = \mathcal{L}(x,v,t) - c \cdot v$ $\qquad c \in \mathbb{R}^d \cong H^1(\mathbb{T}^d, \mathbb{R})$

this is a Galilean boost $\qquad$ first cohomology group

Define h_c^n using $\mathcal{L}_c$



$\frac{1}{n}\mathbb{Z}^d$

Example $\mathcal{L}(x,v,t) = \frac{1}{2}v^2$

$\mathcal{L}_c = \frac{1}{2}(v-c)^2 - \frac{1}{2}c^2$

⑤ Larger Invariant Sets

Define $h_c(x,y) = \liminf_{n\to\infty} \left(h_c^n(x,y) + n\,\alpha(c) \right)$

$\alpha(c) \in \mathbb{R}$ Mañe's Critical Value - to keep liminf from being $\pm\infty$

Def (Aubry Set)

$\mathcal{A}_H(c) = \operatorname{argmin} h_c(x,x) = \{x : h_c(x,x) = 0\}$ $\qquad$ this gives you a periodic orbit starting at x with initial velocity c

$\tilde{\mathcal{A}}_H(c) = \{(x, \partial_2 h_c(x,x)) : x \in \mathcal{A}_H(c)\}$ $\qquad$ family of periodic orbits in (x,p).

compact, invariant under Φ_H^1

$\tilde{\mathcal{A}}$ is Aubry Set, $\mathcal{A}$ is Aubry set projected onto configuration space

Relation to weak KAM:

Aubry set is contained in

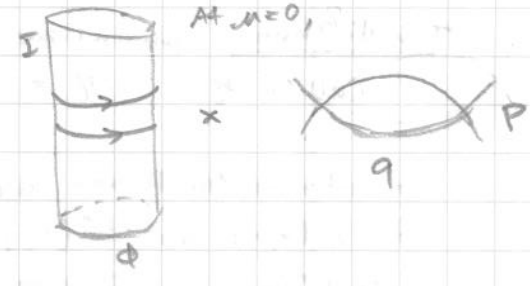
$\tilde{\mathcal{A}}_H(c) \subset \{(x, c + du(x)),$ u is any weak KAM solution of $L_c\}$

Examples

(A) Arnold's Ex. $\quad H_\mu(q,p,\varphi,I,t) = \frac{1}{2}p^2 + (\cos q - 1)(1 - \mu(\cos\varphi + \cos t)) + \frac{1}{2}I^2$

$\tilde{\mathcal{A}}_H((0,c_2)) = \left\{ \begin{matrix} q = p = 0 \\ \varphi \in \mathbb{T},\ I = c_2 \end{matrix} \right\}$ global min even if $\mu \neq 0$

At $\mu = 0$,



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(B) Birkhoff's Ex. $H(q,p,t)$ $q \in \mathbb{T}$, $p \in \mathbb{R}$, $t \in \mathbb{T}$
twist map on a cylinder

$\tilde{\mathcal{A}}_H(c)$ can be:
periodic
invariant circle
Cantor sets

lots of invariant sets found this way

• Mañe Sets and homoclinics to $\tilde{\mathcal{A}}$

$$\tilde{N}_H(c) = \left\{ (x_0, \partial_2 h_c(x_1, x_0)) : \begin{array}{l} x_0 \in \underset{x \in \mathbb{T}^d}{\operatorname{argmin}} \left(h_c(x_1,x) + h_c(x,x_2) \right) \\ x_1, x_2 \in \mathcal{A}_H(c) \end{array} \right\}$$

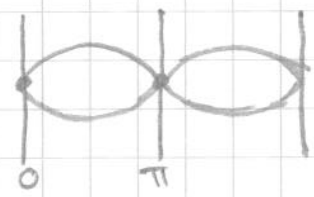
We get a larger invariant set called Mañe.

$\tilde{N}_H(c) \supset \tilde{A}_H(c)$

$\tilde{N}_H(c) \setminus \tilde{A}_H(c) =$ homoclinics to $\tilde{\mathcal{A}}_H(c)$

"true homoclinics"

Arnold Example: $\hat{H} = H_n(2q, \frac{p}{2}, \varphi, I, t)$



Thm (Mather 93) Mather Mechanism (MM)

If you project the Mañe into $\mathbb{T}^d$, suppose set is contractable,
$\pi_x \tilde{N}_H(c) \subset \mathbb{T}^d$
then $\tilde{\mathcal{A}}_H(c)$ and $\tilde{\mathcal{A}}_H(c')$ are heteroclinically connected if $\|c - c'\| \ll 1$

For $d=1$, ($\pi_x \tilde{N}_H(c)$ is contractable) $\approx$ ($\tilde{\mathcal{A}}_H(c)$ is not an invariant curve)
diffusion only happens away from invariant curves.

Thm (Mather 93, Cheng-Yan 04, 09, Bernard 08)

Suppose $\tilde{N}_{\hat{H}}(c) \setminus \tilde{\mathcal{A}}_{\hat{H}(c)}$ consists of isolated points.
Then the same conclusion holds.

Arnold: perturb the map
stable & unstable manifolds intersect transversally
so it has to intersect something nearby

Similar result as yesterday, but totally different method.
Doesn't have to be on a manifold - just closed, bounded.

Thm (Bernard, Kaloshin, Zhang 16)

$$H_\varepsilon(q,p,\phi,I,t) = \tfrac{1}{2}p^2 + \varepsilon(\cos q - 1) + \varepsilon\delta R + \tfrac{1}{2}I^2$$

and $\delta \ll 1$ fixed

Then $\exists$ residual set $\|R\|_{C^2} < 1$ such that $\forall \varepsilon \in (0, \varepsilon_0)$

$\tilde{\mathcal{A}}_H((0,c_2))$ and $\tilde{\mathcal{A}}_H((0,c_2'))$ are connected $\forall c_2, c_2'$

Q How could you find these sets for more complicated dynamical systems?
perhaps numerically find weak Hamilton – Jacobi solutions