Introduction to Weak KAM Theory

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Concerns weak solutions of Hamilton-Jacobi equations

I. Definitions and Assumptions

Notation: $\mathbb{R}^N \ni x$, $t \mapsto x(t)$, $\dot{x}$ deriv with respect to t

$\ddot{x}$ 2^{nd} deriv with respect to t

$\|\cdot\|$ usual Euclidean norm

∇ gradient

Historical Example Celestial Mechanics

Newton's equations:

$m\ddot{x} = \nabla U(x)$ U = potential function m mass

$U : \mathbb{R}^N \to \mathbb{R}$

If $L(x,v) = \frac{1}{2} m \|v\|^2 + U(x)$ is the Lagrangian,

we can rewrite Newton's equations as:

Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x,\dot{x})\right) = \frac{\partial L}{\partial x}(x,\dot{x}) \qquad \text{(EL)}$$

Subexample Free particle, $U = 0$

$m\ddot{x} = 0$, $x(t) = x_0 + t v_0$ ← initial speed

↑ initial position

look at both (position, speed) (2^{nd} order diff eq ⇒ needs two initial conditions

$$\begin{cases} \dot{x} = v \\ m\dot{v} = \nabla U(x) \end{cases} \iff \begin{cases} \dot{x} = v \\ \frac{d}{dt}\left(\frac{\partial L}{\partial v}(x,v)\right) = \frac{\partial L}{\partial x}(x,v) \end{cases}$$

we will need to add some additional hypotheses on L – soon.

Flow "follow the solution with initial (position, speed) = (x,v)"

$\phi_t^L : \mathbb{R}^N \times \mathbb{R}^N \circlearrowleft$ $\phi_t^L(x,v) = (x(t), v(t))$

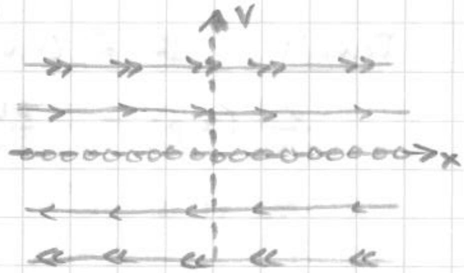
map of position & speed to itself

Subex $\phi_t^L(x,v) = (x+tv, v)$ We've partitioned (x,v) space into subsets where v = constant

Remark Invariant by the flow subsets:

$\{(x,v),\ v = v_0\}$

For $U = 0$



Question of Weak KAM theory

What happens if $N > 1$ and $U \neq 0$?

In general, you don't have an invariant, smooth foliation as for $U = 0$.

→ in a x-periodic case, perturbative, apply (strong KAM)

→ what happens in general?

Problem of Underdetermination

we want the Lagrangian to determine the flow → add hypotheses

Def $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ $\quad C^2 \quad$ = lagrangian

- L is non-degenerate if everywhere, $\det\left(\frac{\partial^2 L}{\partial v^2}\right) \neq 0$.
- the Legendre transform is the C^1 map

$$\mathcal{L}_L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times (\mathbb{R}^N)^*$$
$$(x,v) \mapsto \left(x, \frac{\partial L}{\partial v}(x,v)\right)$$

Remark this is a local diffeomorphism when L is non-degenerate

Remarks

when L is non-degenerate, (EL) can be rewritten as follows:

$$\frac{\partial^2 L}{\partial v^2}\left(x(t), \dot{x}(t)\right) \cdot \ddot{x}(t) = \frac{\partial L}{\partial x}\left(x(t), \dot{x}(t)\right) - \frac{\partial^2 L}{\partial x \partial v}\left(x(t), \dot{x}(t)\right) \cdot \dot{x}(t)$$

no underdetermination

- if $\mathcal{L}_L$ is a C^1 diffeomorphism onto $\mathbb{R}^N \times (\mathbb{R}^N)^*$

define new coordinates:

$$(x,p) = \mathcal{L}_L(x,v)$$

define a new flow $\varphi_t = \mathcal{L}_L \circ f_t \circ \mathcal{L}_L^{-1}$

define the Hamiltonian $\quad H : \mathbb{R}^N \times (\mathbb{R}^N)^* \to \mathbb{R}$

$$H(x,p) = p \cdot v - L(x,v) \quad \text{where} \quad (x,p) = \mathcal{L}_L(x,v)$$

in coordinates (x,p), E-L equations become Hamilton's equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x,p) \\ \dot{p} = -\frac{\partial H}{\partial x}(x,p) \end{cases}$$

(a priori, it appears as though H is C^1

Remark If $\mathcal{L}_L(x,v) = (x,p)$,

$$\frac{\partial H}{\partial x}(x,p) = -\frac{\partial L}{\partial x}(x,v), \quad \frac{\partial H}{\partial p}(x,p) = v, \quad \frac{\partial L}{\partial v}(x,v) = p$$

← Everything here is C^1.

So H is C^2

<u>Proposition</u> Assume that $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is C^2,
is a Lagrangian that is

- C^2 convex in the vertical direction, i.e. $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite
- L is superlinear in the vertical direction, i.e. $\lim_{\|v\| \to +\infty} \frac{L(x,v)}{\|v\|} = +\infty$

then $\mathcal{L}_L$ is a C^1 diffeomorphism onto $\mathbb{R}^N \times (\mathbb{R}^N)^*$
the Hamiltonian satisfies the same hypotheses:
C^2 convex and superlinear in the vertical direction

moreover (1) Young Inequality
$\forall (x,v,p)$ (don't assume $\mathcal{L}_L(x,v) = (x,p)$)
$L(x,v) + H(x,p) \geq p \cdot v$
with $=$ iff $\mathcal{L}_L(x,v) = (x,p)$

(2) $H(x,p) = \max_{v \in \mathbb{R}^N} \left(p \cdot v - L(x,v) \right)$

<u>Example</u> $L(x,v) = \frac{m}{2}\|v\|^2 + U(x)$

$\mathcal{L}_L(x,v) = (x, mv)$ this is a diffeomorphism

$H(x,p) = \underbrace{\frac{\|p\|^2}{2m}}_{\text{Kinetic Energy}} \underbrace{- U(x)}_{\text{Potential Energy}}$

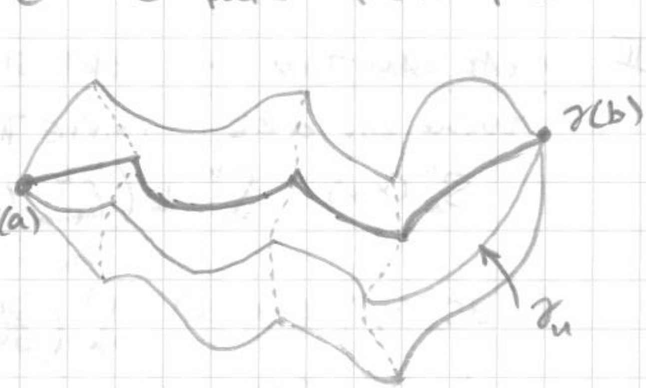
<u>Def</u> L (or H) are Tonelli
if C^2-convexity and superlinearity in the vertical direction

II. <u>Lagrangian Action</u>

$L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is C^2, Tonelli (uses minimization in a variational setting)

<u>Def</u> If $\gamma : [a,b] \to \mathbb{R}^N$ C^0 + piecewise C^1 ← paths in position space

- its action is $A_L(\gamma) = \int_a^b L(\gamma, \dot\gamma)$
- variation of γ



$\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to \mathbb{R}^N$
$(u,t) \mapsto \gamma_u(t)$

- $\forall u,\ \gamma_{u|\{a,b\}} = \gamma_{|\{a,b\}}$ • $\gamma_0 = \gamma$
- Γ and $\frac{\partial \Gamma}{\partial u}$ are C^0
- $\exists$ a subdivision $a = \tau_0 < \cdots < \tau_k = b$ such that on $(-\varepsilon, \varepsilon) \times [\tau_i, \tau_{i+1}]$,

$\frac{\partial^2 \Gamma}{\partial u \partial t} = \frac{\partial^2 \Gamma}{\partial t \partial u}$ is C^0

for example, $\gamma_u(t) = \gamma(t) + u\gamma_1(t)$

$\gamma_1(a) = \gamma_1(b) = 0$, γ_1 is C^0 and piecewise C^1

γ is an extremal of A_L if

for every variation $\frac{\partial}{\partial u}\left(A_L(\Gamma_u)\right)_{u=0} = 0$

<u>Proposition</u> extremal $\Leftrightarrow$ solution of E-L equation

<u>Assumption</u> $\mathbb{Z}^N$ - periodicity (N - torus)

$L(x+k, v) = L(x,v) \quad \forall k \in \mathbb{Z}^N$

$\Rightarrow$ existence of minimizers of the action with fixed ends and they are extremals

<u>Q</u> What would be different on a sphere instead of a torus?

From now on, we assume the Lagrangian is Tonelli & periodic in the x-variable.

<u>Then</u>

For every $x, y \in \mathbb{T}^N$ and $a < b \in \mathbb{R}$

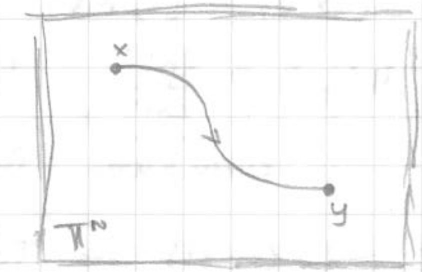
there exists an absolutely continuous path

(we no longer have piecewise C^1)

$\gamma : [a,b] \to \mathbb{T}^N$ such that $\gamma(a) = x$, $\gamma(b) = y$

which minimizes the Lagrangian action.

Weierstrauss satisfies E-L and is C^2.



$\gamma : \underbrace{[a,b]}_{\text{time interval}} \to \mathbb{T}^N$

Lagrangian action: $a_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma})\, dt$

Define a new action between two points for a given time:

$$a(x,y;t) = \min_{\substack{\gamma(0)=x \\ \gamma(t)=y}} a_L(\gamma) \quad \text{for } t>0, \; x,y \in \mathbb{T}^N$$

III WEAK KAM THM - LAGRANGIAN POINT OF VIEW

Define for $a<b$, $(x,v) \in \mathbb{T}^N \times \mathbb{R}^N$ an action:

$$S_a^b(x,v) = \int_a^b L\Big(\underbrace{\phi_{t-a}^L(x,v)}_{\text{this is some } x(t),\, \dot{x}(t) \text{ where } x \text{ solves E-L}}\Big)\, dt$$

$$dS_a^b(x,v)\,(\delta x, \delta v) = \int_a^b \left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t))\, \delta x(t) + \frac{\partial L}{\partial v}(x(t), \dot{x}(t))\, \delta \dot{x}(t) \right) dt$$

integrate by parts

(where $(\delta x(t), \delta \dot{x}(t)) = D\phi_{t-a}^L(x,v)\,(\delta x, \delta v)$.

$$= \left[\frac{\partial L}{\partial v}(x(t), \dot{x}(t))\, \delta x(t) \right]_a^b + \int_a^b \left(\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(x,\dot{x})\right) \right) \delta x(t)\, dt$$

$= p(b)\,\delta x(b) - p(a)\,\delta x(a)$

if $(x(t), p(t)) = \mathcal{L}_L(x(t), \dot{x}(t)) = \left(x(t), \frac{\partial L}{\partial v}(x(t), \dot{x}(t)) \right)$

$$p(b)\,\delta x(b) = \underbrace{p(a)}_{du_a(x(a))}\,\delta x(a) + dS_a^b(x,v)(\delta x, \delta v)$$

if the image by the flow of the graph (du_a) is also a graph, it is the graph of du_b.

with $u_b(x) = u_a(x) + S_a^b(x,v) + C$

Def The Lax-Oleinik semi group

$T^t \; C^0(\mathbb{T}^n, \mathbb{R}) \circlearrowleft$ is defined by

$$T^t_u(x_0) = \operatorname*{Min}_{x \in \mathbb{T}^n} \left(u(x) + a(x, x_0; t) \right)$$

This describes how a graph u is changed by the flow

Properties

- semigroup (only positive t)
- nonexpansiveness
- order preserving
- addition of constant not important
- continuity with respect to t

Thm Weak KAM, part 1

there is a fixed point of the weak semi-group

the fixed point is called the weak KAM solution defined up to a constant C

Idea of Proof

addition of constant not important

work in $C^0(\mathbb{T}^n, \mathbb{R}) / \mathbb{R}$

$$\|u\| = \operatorname*{Min}_{a \in \mathbb{R}} \|u - a\|_\infty$$

non expansive $\to$ contracting

prove fixed point for each t, then prove a fixed curve for all t.

Thm Weak KAM, part 2

- Domination Property:

$$u(x) = \operatorname*{Min}_{\gamma(t) = x} \left(u(\gamma(0)) + \int_0^t (L(\gamma, \dot{\gamma}) + c) \right)$$

$\Rightarrow$ if $\gamma : [0,t] \to \mathbb{T}^N$, then $u(\gamma(t)) - u(\gamma(0)) + \int_0^t (L(\gamma, \dot{\gamma}) + c)$

- We have a solution on the negative interval

solution is γ_k

Solution may not be unique

Solution of E-L since it is minimizing on $(-\infty, 0]$

- c is unique

$c = c[0]$ is Mañé critical value

$c = -\inf_{\mu} \int L \, d\mu$, μ is a Borel invariant probably by ϕ_t^L

<u>Idea of Proof</u>

$$u\left(\pi \circ \phi^t(x,v)\right) - u\left(\pi(x,v)\right) \leq \int_0^t \left(L(\phi^s(x,v)) + c\right) ds$$

projection on $\mathbb{T}^N$ - space

integrate with respect to μ

$0 \leq \int_0^t \left(\int L + c\right) d\mu$

$\Rightarrow -t \int L \, d\mu \leq c t$

To get equality, work with γ_x (calibrated curve) from the second part of the thm.

<u>Q</u> Why is this called the weak KAM thm?

KAM thm:

when you perturb an integrable Lagrangian system, some tori break & some survive. — these are KAM tori

the KAM tori will be graphs of u. – when they exist

<u>IV</u> HAMILTONIAN POINT OF VIEW

If the weak KAM solution u_- is differentiable at $x \in \mathbb{T}^N$, then

- it's in a Hamiltonian level set $H(x, du_-(x)) = c$
- the calibrated curve satisfies the dynamics $du_-(x) = \frac{\partial L}{\partial v}(x, \dot{\gamma}_x(0))$

<u>Hamilton - Jacobi Equation</u>

$H(x, du(x)) = c$ satisfies HJE

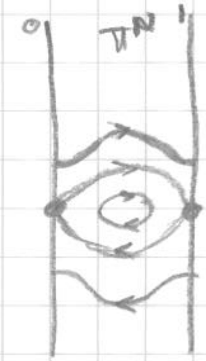
classical solution of HJE is when solution is C^1

<u>Thm</u> u is weak KAM solution $\iff$ $H(x, du(x)) = c$ is a classical solution of HJE

graph of du is an invariant set

Problem: often, we don't have classical solutions of HJE

Ex Pendulum $H(x,p) = \frac{1}{2}p^2 + \varepsilon \cos(2\pi x)$



No classical solution – problems with separatrix.

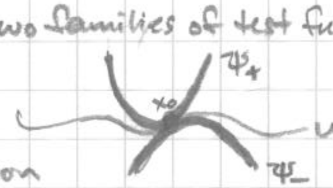
Very Weak Solution of HJE

$u : \mathbb{T}^N \to \mathbb{R}$ is Lipschitz

$H(x, du(x)) = c$ when it exists

this is too weak – many such solutions

Def $u : \mathbb{T}^N \to \mathbb{R}$ is a viscosity solution of HJE at x_0
if for every ψ_+ (ψ_-) $\mathbb{T}^N \to \mathbb{R}$ C^1 (two families of test functions)
such that · $\psi_+ \geq u$ $(\psi_- \leq u)$
· $\psi_+(x_0) = u(x_0)$ $(\psi_-(x_0) = u(x_0))$
then $H(x_0, d\psi_+(x_0)) \leq c$ is the viscosity subsolution
$H(x_0, d\psi_-(x_0)) \geq c$ is the viscosity supersolution

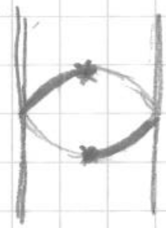


this def must hold for each x_0.

The classical solution is always a viscosity solution

– looking for weak KAM solution

Pendulum at separatrix
graph of du is:
backward invariant
integrates to zero



Several theorems on slides

Regularity Thm

C^1 weak solution is $C^{1,1}$ – du is Lipschitz

Thm

if u is a classical solution of HJE, dynamics restricted to graph of du
is Lipschitz conjugate to a rotation
the u is C^2

Q Intuition for near-integrable systems?

Even if you don't have a foliation by KAM tori,
you have graphs of du.

Q Can you obtain a family of solutions?

modify Lagrangian $L + \lambda_0 v$ – doesn't modify dynamics,
but modifies minimizers