9A

PERIODIC ORBITS OF TONELLI LAGRANGIAN SYSTEMS

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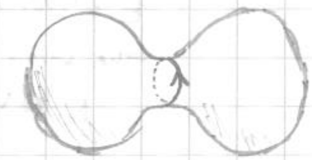
Typical example - geodesic flows - periodic orbits are closed geodesics

Closed Geodesics



Poincaré (S^2, g)

on any sphere, there is a closed geodesic without self-intersections

not completely proved until 1980's

no restrictions on the metric

Birkhoff - Lusternik - Fet

there exist a closed geodesic on every closed manifold (M, g)

Gromoll - Meyer 69 + Vigué-Poirier Sullivan

for any closed manifold M, $\pi_1(M)=0$ with $H^*(M;\mathbb{Q}) \ncong \frac{\mathbb{Q}[x]}{(x^m)}$

$\forall g$, there exist infinitely many closed geodesics

Bangert + Franks + Hingston

Infinitely many closed geodesics on every Riemannian sphere (S^2, g)

Conjecture

every closed Riemannian manifold of dimension at least 2 (M^n, g) $n \geq 2$

has infinitely many closed geodesics

VI often, spheres are hard enough - replace M^n with S^n

Conjecture

every closed (S^n, g), $n \geq 5$ has at least 2 closed geodesics

there are results for $n = 3, 4$.

the various cases that have been proved are by very different means

↳ there may be weird counterexamples.

Intermediate Conjecture

If all closed geodesics on (S^n, g) have the same length ℓ, then (S^n, g) is Zoll

Zoll := every geodesic is closed with length ℓ.

Thm in $n=2$ (Mazzucchelli - Suhr Lusternik)

what if we use Finsler metric F instead of Riemannian metric g?

Poincaré - unknown

BLF, GM + PS - still holds

BFH does NOT hold for (S^2, F) - counterexample by Katok

TONELLI LAGR / HAM

M closed

$L: TM \to \mathbb{R}$ Tonelli Lagr $\xleftrightarrow{\text{dual}}$ $H: T^*M \to \mathbb{R}$

$L(q,v) + H(q,p) = pv$ $\quad \begin{cases} p = \partial_v L(q,v) \\ v = \partial_p H(q,p) \end{cases}$

for some fixed energy $e \in \mathbb{R}$
the flow stays in its level set
$H^{-1}(e) \circlearrowleft \Phi_H^t$

energy in a tangent bundle:

$E: TM \to \mathbb{R}$ $\quad E(q,v) = \partial_v L(q,v) - v\,\partial_v L(q,v)$

the flow stays in this level set

$E^{-1}(e) \circlearrowleft \Phi_L^t$ $\quad \Phi_L^t(\gamma(0), \dot\gamma(0)) = (\gamma(t), \dot\gamma(t))$

where $\gamma: \mathbb{R} \to M$ satisfies Euler-Lagrange Eqtns.

Ex $L(q,v) = \frac{1}{2}\|v\|_g^2 - \Theta_q(v)$ $\quad \Theta = $ 1-form on M not closed

$E(q,v) = \frac{1}{2}\|v\|^2$

$H(q,p) = \frac{1}{2}\|v + \Theta_q\|^2$

$\Phi_L^t(\gamma(0), \dot\gamma(0)) = (\gamma(t), \dot\gamma(t))$ where $\frac{\nabla}{dt}\dot\gamma = Y(\dot\gamma)$ (covariant derivative)

$g(Y(v), w) = d\Theta(v,w)$

Called: Magnetic Geodesic Flow. — this is exact

Q How many periodic orbits are there on any given energy hypersurface? $E^{-1}(e) \circlearrowleft \Phi_L^t$

An orbit $(\gamma, \dot\gamma): \mathbb{R} \to TM$ is a

τ - periodic orbit of the flow of the hypersurface $E^{-1}(e) \circlearrowleft \Phi_L^t$

iff the orbit minimizes the action $\quad \gamma \in \mathrm{Crit}(S_e)$

action is a functional $S_e : \underset{\text{time}}{(0,\infty)} \times \underset{\text{periodic curve}}{W^{1,2}(\mathbb{R}/\mathbb{Z}, M)} \to \mathbb{R}$

free-period

$(\tau, \Gamma) := \gamma \quad \gamma(t) = \Gamma(t/\tau)$

$S_e(\gamma) = \int_0^\tau L(\gamma, \dot\gamma)\,dt + \tau e$

Connection to Weak KAM:

$H(q, du(q)) = c(L)$ Mañé critical value

$c(L) := \inf\{e \in \mathbb{R} \mid S_e \geq 0\}$

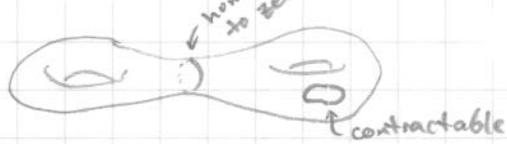
marks the difference between low energy & high energy dynamics

$\tilde M$ universal cover
$\downarrow$
M_0 universal abelian cover $\quad$ covering space is $\pi_1(M_0) = [\pi_1(M), \pi_1(M)]$
$\downarrow$
M

lifts of the Lagrangian: $L_0 : TM_0 \to \mathbb{R}$, $\tilde L : T\tilde M \to \mathbb{R}$

$c(L) \geq c(L_0) \geq c(\tilde L) \geq e_0(L)$

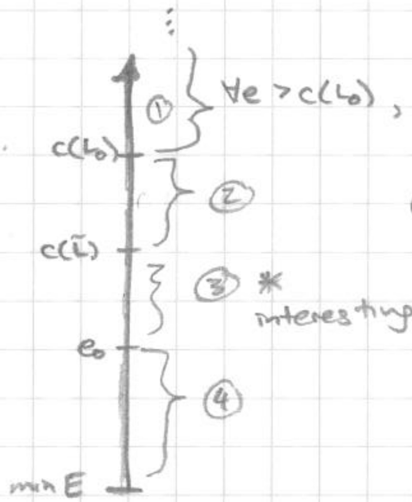
$c(L_0) = \inf\left\{e \;\middle|\; \begin{array}{l} S_e(\gamma) \geq 0 \\ \forall [\gamma] = 0 \text{ in } H_1(M) \end{array}\right\}$



$c(\tilde L) = \inf\{e \mid S_e(\gamma) \geq 0 \;\; \forall \text{ contractable } \gamma\}$

$c(H) = \inf_u \max_q H(q, du(q))$

$e_0(L) = \inf\{e \mid E^{-1}(e) \twoheadrightarrow M\}$
below this, the particles are trapped.



$\exists!$ Finsler metric F_e s.t. $E^{-1}(e) \circlearrowleft \phi_L^t$ is a geodesic flow of F_e
(Contreras - Iturriaga - Paternain²)

① for high enough energy, the flow is a geodesic flow with a Finsler metric.

② S_e is as good as the Finsler metric
this might be trivial if $c(L_0) = c(\tilde{L})$

④ Energy hypersurface doesn't project down to the whole manifold
$\forall e \in (\min E, e_0)$
$H^{-1}(e) \subset T^*M$ is Hamiltonian displaceable
for all of these e's, there exists a periodic orbit (Contreras, Frauen-Schlenk)

③ for $L(q,v) = \|v\|^2 - \Theta_q(v)$, if $d\Theta \not\equiv 0$, then $e < c(\tilde{L})$
so this interval is nonempty.

S_e has bad properties:
unbounded from below - can't find periodic orbits by minimizing S_e
no Palais - Smale - don't have compactness to use max/min

Contreras
for almost all $e \in (e_0, c(\tilde{L}))$, there exists a periodic orbit γ with $S_e(\gamma) > 0$

This is what is known in general setting

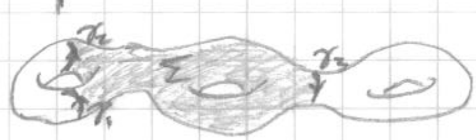
Much more is known in dim-2, 2-orientable

If a curve has a self-intersection, it cannot be broken by a small perturbation

- Taimanov, Contreras - Macarini - Paternain, Asselle - Benedetti - M.

$\mathbb{B} = \{\underline{\gamma} = \gamma_1 \cup \dots \cup \gamma_n \hookrightarrow M \mid \underline{\gamma} \text{ oriented boundary of } \Sigma \subset M \text{ open } \neq \emptyset\}$

union of closed curves embedded in M



each γ_i is a p.o. with energy e, local min of S_e.

Thm $\exists \underline{\gamma} \in \mathbb{B}$ such that $S_e(\underline{\gamma}) = \inf_{\mathbb{B}} S_e$

Thm (Abbondandolo, Macarini, Mazzucchelli, Paternain)
For almost all $e \in (e_0, c(\tilde{L}))$, there exist infinitely many periodic orbits

open - all?, higher dim?