12A

INTERACTION OF TWO CHARGED PARTICLES IN A UNIFORM MAGNETIC FIELD

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Application of Hamiltonian dynamics to illustrate:

- finding integrals
- second species chaos
- transition states
- transport in plasmas

One charge in a magnetic field

mass m, charge e, position $\underline{q} = (x,y,z)$, velocity $\underline{v}$

field $\underline{B}$, associated flux form $\beta = i_{\underline{B}}\, dx \wedge dy \wedge dz$

β closed $\Leftrightarrow \nabla \cdot \underline{B} = 0$

$H = \frac{1}{2} m |\underline{v}|^2$, $\omega = m\bar{\omega} - e\beta$, with $\bar{\omega} = \sum_i dq_i \wedge dv_i$

Gives vector field $X_H = (\dot{\underline{q}}, \dot{\underline{v}})$ by $i_{X_H}\omega = dH$

i.e. $\omega(X_H, \xi) = dH(\xi)$ $\forall$ tangents ξ

Case: $\underline{B} = B\hat{z}$, $\beta = B\, dx \wedge dy$, write $\underline{v} = (\underline{v}_\perp, v_\parallel)$

Solution is a Larmor helix at gyrofrequency $\Omega = eB/m$ about the field line with any $v_\parallel$, any radius

Gyroradius vector $\underline{\rho} = -\frac{1}{\Omega} J \underline{v}_\perp$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Guiding center $\underline{R} = \underline{q} - \underline{\rho}$

Magnetic moment $\mu = \frac{1}{2} e\Omega \|\underline{\rho}\|^2$

$H = \frac{1}{2} m v_\parallel^2 + \mu B$

Two Charges without a Magnetic Field

$$H = \frac{1}{2}\sum_{j=1}^{2} m_j |\underline{v}^j|^2 + \frac{\kappa e_1 e_2}{|\underline{q}^1 - \underline{q}^2|}, \quad \omega = \sum_j m_j \bar{\omega}^j$$

$\begin{cases} \text{if } e_1 e_2 < 0, & \text{Kepler Problem} \\ \text{if } e_1 e_2 > 0, & \text{Rutherford Scattering} \end{cases}$

Two Charges in a Uniform Magnetic Field

$$H = \frac{1}{2}\sum_{j=1}^{2} m_j |\underline{v}^j|^2 + \frac{\kappa e_1 e_2}{|\underline{q}^1 - \underline{q}^2|}, \quad \omega = \sum_{j=1}^{2} (m_j \bar{\omega}^j - e_j \beta^j)$$

6 degrees of freedom: $(\underline{q}^1, \underline{v}^1, \underline{q}^2, \underline{v}^2)$

Continuous Symmetries: vector field V that preserves H, ω

- translations
- rotations about $x=0, y=0$ (any field line)

100th year anniversary

<u>Noether's Theorem</u>: A continuous symmetry V for (H, ω) implies a conserved quantity Q for the Hamiltonian vector field X_H.

viz. $dQ = i_V \omega$

Vertical Translations $\rightarrow P_{\parallel} = \sum_j m_j v_{\parallel}^j$

Horizontal Translations $\longrightarrow \underset{\sim}{P}_{\perp} = \sum_j \left(m_j \underset{\sim}{v}_{\perp}^j - JB e_j q_{\perp}^j \right) = -JB \sum_j e_j \underset{\sim}{R}_{\perp}^j$

so if $e_1 + e_2 \neq 0$, then the center of charge $\underset{\sim}{R}^* = \sum e_j \underset{\sim}{R}_{\perp}^j / \sum e_j$ is conserved

if $e_1 + e_2 = 0$, then $\underset{\sim}{R}_{\perp}^1 - \underset{\sim}{R}_{\perp}^2$ is conserved

Rotation $\rightarrow L = \sum_j \left(m_j q_{\perp}^j \cdot J v_{\perp}^j + \frac{1}{2} e_j B |q_{\perp}^j|^2 \right)$ angular momentum

$$= \frac{1}{2} B \sum \left(e_j |\underset{\sim}{R}_{\perp}^j|^2 - \frac{\mu_j}{\Omega_j} \right)$$

Say two functions F, G are in involution if their Poisson bracket is zero.

where $\{F, G\} = \omega(X_F, X_G)$

$P_{\parallel}, L, |\underset{\sim}{P}_{\perp}|^2$ are in involution $\Rightarrow$ we can reduce to a 3 degree of freedom system. for each level set of these three functions

Obtain formulae for the reduced system.

If $\Omega_1 = \Omega_2$, there is an additional conserved quantity

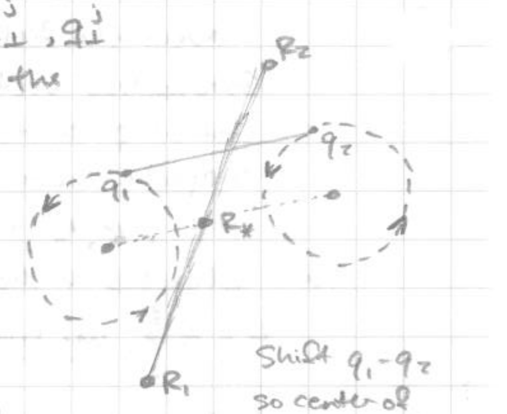
$$W = \left| \sum_j m_j v_{\perp}^j \right|^2 = B^2 \left| \sum_j e_j \underset{\sim}{\rho}^j \right|^2$$

and it is in involution with $P_{\parallel}, L, |\underset{\sim}{P}_{\perp}|^2$ so we can reduce to 2 degrees of freedom

Associated symmetry V is "locomotive coupling rod"

given the horizontal guiding center & horizontal position $\underset{\sim}{R}_{\perp}^j, q_{\perp}^j$

locate $\underset{\sim}{R}^*$, translate $q_{\perp}^1 - q_{\perp}^2$ to vector M with $\underset{\sim}{R}^*$ as the center of charge of its ends.



Shift $q_1 - q_2$ so center of charge is R_*

Symmetry; rotate both around these end points

Coplanar Motion Λ 4D in 6D

case $e_1 e_2 < 0$ $\quad \Omega_1 + \Omega_2 \neq 0$

with high enough energy $\Rightarrow$ "second species chaos"

Poincaré called a periodic orbit of the three body problem with 2 small masses "second species" if as mass $\rightarrow 0$ it limits to a concatenation of pairs of Kepler orbits joined at collisions.

In restricted case, these are fly by case.

Poincaré claimed there exist infinitely many of these.

CR3BP - Proof by Bolotin & MacKay 2000 (planar), 2006 (space)

also made subshifts of chaotic orbits

Two Charges in a magnetic field.

choose 2 orbits (angle, radius) that collide twice.

After 2nd collision, choose new orbits that will collide again.

Repeat $\rightarrow$ sequence

subject to uniformity conditions & high energy $\rightarrow$ get a time orbit close to the sequence of collisions

3D Dynamics with Opposite Signs of Charge

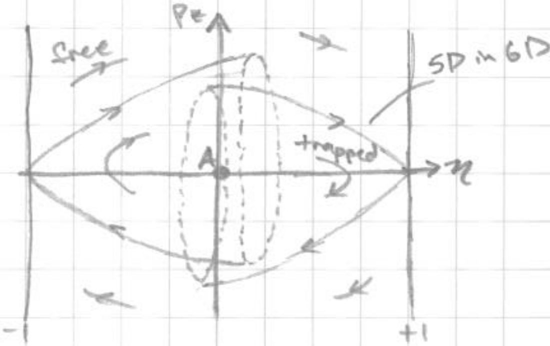
Natural Coords: $q_z = q_z^1 - q_z^2$

$$P_z = \frac{m_1 m_2}{m_1 + m_2}\left(V_z^1 - V_z^2\right)$$

$q_z = 0$ is what we looked at before – regularize around ∞ instead

$$q_z = \frac{\eta}{(1-\eta^2)^2}, \quad \frac{ds}{dt} = \frac{1}{(1-\eta^2)^3}$$

there is sort of an invariant set at ∞ ($\eta = \pm 1$)



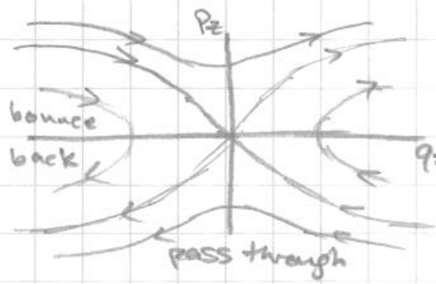
3D dynamics with the same sign of charge:

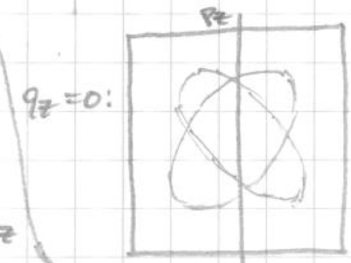
Λ "nominally hyperbolic" – has stable/unstable manifolds (truly if $\Omega_1 = \Omega_2$)

$\Lambda_{E>E'} \cong S^3$

"transition state"

spanned by 2 hemi-S^4 of unidirectional flux





Scattering Map relates $\underline{R}_\perp^j, \mu_j$ before and after

No net transfer of charge – center of charge fixed

But can get energy transport across the magnetic field

Principal effect if guiding centers far apart