Introduction to Periodic Orbits & Invariant Manifolds in Celestial Mechanics with Applications to Space Missions

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Celestial Mechanics

Division:

Natural Bodies

- long & short time scales
- problem: motion of asteroids - hazard mitigation from near-Earth asteroids
- dominated by gravity - maybe solar radiation pressure
- analysis - not control

Spacecraft

- short time scale (up to decade)
- maneuvers / thrust - we impart Δv to the spacecraft
- goal: to design spacecraft orbits

Models:

- circular restricted 3-body model
- four-body models
- elliptic restricted 3-body model
- augmented hill - includes radiation pressure } for asteroids
- shape models }
- ephemeris models - most realistic

All of these are described by a continuous dynamical system.
looking for fixed points, periodic orbits

Two-Body Problem

solved

"patched-conic" method for 3-body problem

circular orbit around sun, then hyperbolic near planet, then elliptic around sun.

Circular-Restricted 3 Body Problem

use a rotating frame with the primary & secondary bodies. - point x along this line

this modifies the Hamiltonian for the body of negligible mass

primary - mass $1-\mu$, located at $-\mu$; secondary - mass μ, located at $1-\mu$

5 equilibrium points - libration points.

Jacobi constant of motion $\nearrow$ $C = x^2+y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$

Tisserand criterion - this would remain about the same for a comet before & after a flyby for a planet.

for most systems, mass ratio μ is very small.

Earth-moon is the largest we study

Rotating Frame

can see much more information

see number of close passes / periapses

can see transitions between orbits much easier

Hill's Regions

look at Jacobi constant - where is velocity real? imaginary?

forbidden region = Hill's regions - where velocity is imaginary

as Jacobi constant ↑ (energy ↓), forbidden regions increase

may close off whether you can get to / past the secondary

often, there's a small tube near the secondary where the satellite can transfer from the inner region to the outer region

Symmetries

can use them to compute periodic orbits

single-shooting algorithm

determine how you need to change the current velocity to get where you want to go at a later time.

search for two perpendicular crossings of the plane left invariant by the action of the symmetry

gives you a symmetric periodic orbit

Periodic Orbits of CRTBP:

- planar Lyapunov orbits, vertical Lyapunov orbits around a lagrange point
  family around each lagrange point
- Halo orbits - halo over the moon
  could be used to relay messages to astronauts on the far side of the moon
  most are unstable; some stable near the moon
- Secondary - Centered Orbits
  different μ, radius of planet affect this significantly
  can have weird shapes - not just near Keplerian.
- Distant retrograde orbits
  these are stable - harder to get to, but spacecraft won't leave.
  can get period doubling bifurcations
- Unstable exterior resonant orbits
  centered around primary; also go around secondary
- Unstable Orbits with Close Flybys
  Arenstorf, Fehlberg
  used to test numerical integrators

Libration Point Quasiperiodic Orbit Methods

looks for quasiperiodic orbits around Lagrange points

refines on an initial guess - multi-shooting differential corrections

initial guess can come from simpler model.

Lissajous Orbits - transition between horizontal & vertical Lyapunov orbits

Quasihalo orbits - quasiperiodic orbits around halo orbits

Cell-Mapping to find Periodic Orbits

more computationally intensive, but can find more interesting orbits

create cell state space - region of interest, sink cell outside

integrate forward for some time

see which cells the orbits end up in - need to correct location here

look for fixed cells, periodic cells, multiply periodic cells

can find interesting, long orbits → different views of surface of secondary

(shadowing thm for fine enough grid to get actual periodic orbit)

Continuation Methods

if you have an orbit in a family, you can change energy / Jacobi constant to find other orbits in same family - similar properties

can use this to look for bifurcation points, create bifurcation maps

Hill's Model

centered around secondary

Four-Body Models

Bicircular - Sun, Earth, Moon - centered on Earth

sun is nonautonomous perturbation

more Lagrange points - often want to transfer moon → Earth Lagrange points

Sun & 2 Planets

Elliptic Restricted 3-Body Models

sometimes, we need to include eccentricity of secondary

Ephemeris Models

N-body of each planet & moon

obtained from observations, integrating full N-body

accurate for a certain epoch

Asteroid Models

as many asteroids as models

asteroid going around sun, strange shape (use spherical harmonics)

we want to send satellite to asteroid - gravity from weird shape, radiation pressure

multiple models depending on how close sattelite is to asteroid

eccentric orbit ⇒ satellite has different radiation pressure

we can find periodic, quasiperiodic orbits in these models

no symmetries - find periodic orbits by looking near equilibrium points

Stability of Periodic Orbits & Invariant Stable/Unstable Manifolds

integrate a variation of the initial conditions for one period
linearized — monodromy matrix
eigenvalues tell you stability   stable if $-1 \le \lambda \le 1$
stable & unstable manifolds for a saddle periodic orbit
  stable - approach orbit as $t \to +\infty$
  unstable - approach orbit as $t \to -\infty$
to find invariant manifolds,
  add a small offset in the stable (unstable) eigenvector's direction and integrate it backward (forward) in time
  do this for each point along the orbit → tube of orbits
can also compute stable & unstable manifolds for the lagrange points.
useful for planning how to transfer to/from orbits
need to look at whether the orbits are accessible from Earth & how long they take
at different energies or Jacobi constant, there are families of nested invariant manifolds
  when you do a burn, it changes the energy & the manifolds

Poincaré Maps

take a surface of section transverse to the flow
often look at resonance orbits
a stable resonant orbit will have quasiperiodic orbits around it

Homoclinic & Heteroclinic Connections

look for them in the surface of section.
homoclinic - goes from unstable to stable manifold in one orbit
heteroclinic - unstable manifold from one orbit connects to stable manifold of another
these are almost free maneuvers - don't need fuel
to find:
  plot stable & unstable manifolds in a surface of section
  the intersections are a connection
    if there are none, maybe do a burn, maybe look at other energies
    if there are some, pick one - maybe we want a close approach, etc.
    if there are multiple intersections on an orbit, it makes the orbit longer
      might have multiple flybys, etc.

Transit & Transport

near libration (Lagrange) points, there are often gaps in between
Hill's forbidden regions.
do analysis here - lots of qualitatively different orbits
using this gateway, we can look at low energy asteroids (slow with respect to Earth)
lots of interesting transfer orbits
Earth-moon with influence of sun.
more challenging problem - find transfer between two orbits at different
energies with minimal Δv
↳ take unstable manifold out, do burn, take stable manifold in

Approach Problem
try to tie into a trajectory that approaches Europa.
follow manifolds of a halo orbit
found originally by brute force numerical search
easier to find them if you know the invariant manifolds
might want to go to Lyapunov orbit first, later transfer to surface
can also tell you where you can land
changing the energy allows you to access more of the surface
the time it takes to reach the surface can also vary dramatically
complex transfers - use a chain of heteroclinic connections to go further

Mission Examples

Lagrange Point Missions

ISEE-3 1978
Lagrange points useful to look at sun since constant relative location
found orbit numerically brute force - actually flew a halo orbit
considered Lissajous orbit - but can't communicate while it's crossing the sun

SOHO 1995

ACE 1997

WIND 1994
lunar flybys, then to Lissajous

MAP 2001
looking at cosmic microwave background - L2 - away from sun
longer to get there

Genesis 2001
first to use dynamical systems theory
required manuevers every 3 months
reentry needed daylight - helicopter grab parachute out of air
heteroclinic to L2, then to Earth

ARTEMIS 2009
first to use Earth-moon libration orbits
extension of a previous mission - not much propellant left
LL2 → LL1 → moon

2 spacecraft
→ two orbits

GRAIL 2011

wanted to arrive at moon at a particular time
regardless of when in the launch period you left
2 spacecraft orbit with precise relative positions

Lunar Trajectory Design Examples

How to go to the moon.

If you're going to use sun to improve trajectory, it takes 3-4 months
Direct, Conventional Transfers
  Apollo - 3 days travel time
various other options
constraints at end - e.g. going to a polar orbit can dramatically change trajectory

weird orbits allow you to reduce cost, achieve specific end states

often, when designing a tour, we want to rotate the orbit's ellipse
  changes lighting condition on surface
  want spacecraft visible to Earth
can do this by changing size of orbit
find a true periodic orbit that looks like this to
  get more accurate description of motion