THE MAXIMAL RANK CONJECTURE AND MODULI OF CURVES

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§1. Complex projective curves and their linear series

Let C be a projective curve of genus g over $\mathbb{C}$, that is, a Riemann surface (a donut with g holes). As C is projective, it can be immersed in $\mathbb{P}^r$ for some r, as zero locus of homogeneous polynomials or equivalently as the intersection of some hypersurfaces. We will think of C together with such an immersion. Consider $C \to \mathbb{P}^r$. By pulling back $\mathcal{O}_{\mathbb{P}^r}(1)$ to C, we get a line bundle on C with $r+1$ independent sections. Conversely, such a line bundle gives a map to $\mathbb{P}^{r+1}$. So thinking about curves in $\mathbb{P}^r$ amounts to considering curves plus line bundles with $r+1$ sections.

1.1. Definition. $G^r_d(C)$ is defined to be the set of linear series of degree d (numbers of zeros of a section or number of points of intersection with a hyperplane) and dimension r on C. In fact, $G^r_d(C)$ is a scheme, which, for generic C, has dimension $\rho := g - (r+1)(g-d+r)$.

Considering the generic curve: if $\rho < 0$, then the locus $G^r_d(C)$ is empty. If $\rho = 0$, consider a rectangular grid of side lengths $r+1$ and $g-d+r$. Then there are g points in the grid. By counting all the ways of filling the rectangle with the numbers 1 throught g in such a way that rows and columns are increasing, we count the number of g^r_d's. If $\rho > 0$, the locus in question is irreducible.

Now consider the Petri map obtained by taking cup product of section

$$H^0(C, L) \otimes H^0(K \otimes L^*) \to H^0(K).$$

Recall that the tangent space to the Picard variety parametrizing line bundles of a fixed degree is given as $H^1(C, \mathcal{O}_C)$. By Sere duality, $H^1(C, \mathcal{O}_C) \simeq H^0(C, K_C)^*$.

The tangent space, $T_{(C,L)}G^r_d$ is identified with the orthogonal to the image of the Petri map. Note that if $\dim H^0(C, L) = r+1$, $\dim H^0(K \otimes L^*) = g-d+r$. Also $\dim H^0(K) = g$. The Brill-Noether number ρ is the difference between the dimensions of the codomain and the domain of the Petri map. If the map is injective, $G^r_d(C)$ is non-singular of the right dimension at the point. The Petri map is actually injective for the generic curve and every line bundle.

Note that all this is for generic curves. In general, considering the curves for which the dimension of G^r_d is larger than expected gives us special loci in the moduli space of genus g curves.

§2. Maximal rank conjectures

Question: how do quadrics, cubics, quartics and other higher-degree hypersurfaces in $\mathbb{P}^r$ cut the curve?

This is addressed by

2.1. Conjecture. *(see* [H]*) The maximal rank conjecture (proposed by Harris in the 1980s): for C generic, and L a generic line budle on C giving rise to $C \to \mathbb{P}^r$ of degree d (meaning generic in $G^r_d(C)$). Then the map*

$$S^kH^0(L) \to H^0(C, L^k)$$

has maximal possible rank. (Here, S^k denotes the k-th symmetric power.

2.2. Remark. This is different for the requirement for the Petri map – the curve and line bundle are both generic, rather than having to check every line bundle on a generic curve. So a priori one might think that the maximal rank conjecture is easier. However, a proof was much more elusive.

The maximal rank conjecture is now proved. The case $k = 2$ (and several other special cases) was first proved by Ballico in [B] (see also [BE], [BE2]). They looked at degenerate curves in the Hilbert Scheme.

Recent proofs of the case $k = 2$ (and some extensions) have been given independently by Liu–Osserman–Zhang–Teixidor ([LOTZ]) and by Jensen–Payne ([JP]). Both groups of authors also proved some cases for higher k with additional conditions. They used new proof methods looking not for a small kernel but rather for a large image in the map and considering curves in an intrinsic way rather as points of a Hilbert scheme.

A complete proof for all k has been given by Eric Larson [L], [L2], [L3].

We now want to look at what happens for every line bundle

2.3. Proposition. [T] *For C a general curve of genus g and L ANY line bundle of degree at most $(g + 1)$, the map*

$$S^2H^0(L) \to H^0(L^2) \tag{1}$$

is injective.

Method of proof. Note that the condition that this map be injective is open (think of it as the complement of a locus cut out by finitely many minors of a matrix). So, if we can find a curve for which this holds, then it is true in a neighborhood of the curve. The strategy is to find/ produce a curve for which the fact holds.

We will use chains of elliptic curves. We need to figure out what happens to linear series that are limits of series on smooth curves. If we degenerate smooth curves to the chain of elliptic ones and degenerate line bundles on the smooth curve, we do not have a unique limit line bundle on the chain. We could modify a line bundle on the family by adding to the corresponding divisor something supported on the special curve. This gives us a lot of choices for line bundles restricted to the special curve. The spaces of sections for the different line bundles are related to each other. Assume that there is a section in the kernel of the map. One checks then that the vanishing at a point at the start of the chain is at least 2. When moving from one elliptic curve to the next, the order of vanishing increases in at least two. This implies that the vanishing at the end of the chain is larger than 2g+2 which is incompatible with the degree of the line bundle.

Moreover, the bound on the degree is best possible: As $\rho(g, 1\frac{g+2}{2}) = 0$, on a generic curve of even genus $G^1_{\frac{g+2}{2}} \neq \emptyset$ consists of several points. We take $L_1, L_2 \in G^1_{\frac{g+2}{2}}$. Write s_1, t_1 (resp s_2, t_2) for independent sections of L_1, L_2. Define L as $L = L_1 L_2 \in G^3_{g+2}$. Then

$$s_1 s_2 \otimes t_1 t_2 + t_1 t_2 \otimes s_1 s_2 - (s_1 t_2 \otimes t_1 s_2 + t_1 s_2 \otimes s_1 t_2) \in \ker[S^2 H^0(L) \to H^0(L^2)]$$

□

§3. Strong maximal rank conjecture, $k = 2$

Farkas–Aprodu [AF] proposed the strong maximal rank conjecture:

3.1. Conjecture (Strong maximal rank conjecture). *For C generic, consider the map $S^2 H^0(L) \to H^0(L^2)$ as a map of vector bundles over $G^r_d(C)$ (a scheme of dimension ρ). Then the dimension of the locus in $G^r_d(C)$ for which this map has non-maximal rank is*

$$\alpha := \rho - \Big(2d + 1 - g - [\binom{r+2}{2} - 1]\Big).$$

Applications of the maximal rank conjecture (suggested by Farkas [F]) have to do with understanding the moduli space $\mathcal{M}_g$ of genus g curves. There are many compactifications for this moduli space, but the simpler is $\overline{\mathcal{M}}_g$, the compatification by stable nodal curves. Here, "stable" means not too many automorphisms.

The classical assumption is that all such $\overline{\mathcal{M}}_g$ are rational, given low genus behavior:

- For $g = 0$, there's only one curve of genus 0
- For $g = 1$, we can parameterize curves as $y^2 = x(x-1)(x-t)$ we have a line
- For $g = 2$, we can parameterize the moduli space by 6 points up to automorphiss of $\mathbb{P}^1$.
- For $g = 3$ most curves are plane quadrics and can be parameterized by the coefficients of it equation modulo automorphisms of $\mathbb{P}^2$.
- ...

But we now know that not all $\overline{\mathcal{M}_g}$ are rational or even unirational (the latter meaning that an open set of the moduli space is mapped onto by some $\mathbb{P}^r$).

Up to $g = 13$, all $\overline{\mathcal{M}_g}$ are in fact unirational.

For $g \geq 24$, it's known that the moduli spaces are "as far from rational as possible." What does this mean? That the schemes are **of general type.** (Due to Harris–Mumford–Eisenbud.)

3.2. Definition. For X of dimension n, nonsingular or with mild singularities, consider $\wedge^n T^*_X = K_X$, the canonical bundle. Taking powers of K, we get a map to $\mathbb{P}^r$. The **Kodaira dimension** of X is defined as follows:

- If multiples of K have no sections, then we say it is $-\infty$.
- Otherwise, the Kodaira dimension $\dim_K X$ is defined as the maximum possible dimension of the image of X by the map corresponding to the linear series $|mK_X|$ as $m \to \infty$.

We say that X is **of general type** if the Kodaira dimension is the maximal possible: the dimension of X itself.

For $g \geq 24$, $\dim_K \overline{\mathcal{M}_g} = 3g - 3$ (dee [HM], [H2], [EH]).

The approach to showing of general type: write down an ample divisor on $\overline{\mathcal{M}_g}$ and relate it to the canonical. We can write down the canonical class explicitly as follows:

$$K_{\overline{\mathcal{M}_g}} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor} \tag{2}$$

3.3. Remark. $\mathrm{Pic}(\overline{\mathcal{M}_g})$ is spanned by λ, arising from the Hodge bundle, and boundary classes δ_i, $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$ corresponding to types of singular curves in $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$. For $i = 0$, we are talking of irreducible curves with one node. For $i = 1, \ldots$ they are reducible curves with one component of genus i and one of genus $g - i$.

To show a multiple of the canonical is ample, it suffices to find an effective divisor D such that

$$D = a\lambda - (\sum b_i \delta_i),$$

with a, $b_i \geq 0$ and

$$\frac{a}{b_i} < \frac{13}{2},$$
$$\frac{a}{b_1} < \frac{13}{3}.$$

The divisors used for $g \geq 24$ came from Brill Noether Theory. When $\rho(g, r, d) < 0$, the generic curve of genus g does not have a linear series of degree d and dimension r. The expectation is that the locus of curves that have such a linear series has codimension $-\rho$ in $\overline{\mathcal{M}_g}$. In fact, for $\rho = -1$, this is known to be true. So, for instance, for odd g, one can choose a suitable d such that for $r = 1$ gives $-1 = g - 2(g - d + 1)$. For g even, things are more involved. The locus of curves where the Petri map is not injective for certain degrees was used too.

These divisors can be defined for every genus, the problem is that for $g < 24$ the coefficients a, b_i do not satisfy the right inequalities.

Farkas' proposal was to consider the locus of curves where the maximal rank conjecture fails to obtain such a divisor. For $g = 22$, $g = 23$, this approach can almost be made to work.

Consider G^6_{25} in $\overline{\mathcal{M}_{22}}$ and $G^6_2 6$ in $\overline{\mathcal{M}_{23}}$, respectively. The locus in these where the appropriate map (1) fails to be maximal rank gives divisors on the relevant G^r_d's. We can compute their classes and their coefficients satisfy the right inequalities.

Problem:

- We need the strong maximal rank conjecture for this to work , otherwise, these loci could be the whole $\overline{\mathcal{M}_{22}}$, $\overline{\mathcal{M}_{23}}$.

- The computation of the classes are carried out in $\mathcal{G}^6_{25}$ lying over $\overline{\mathcal{M}_{22}}$ and $\mathcal{G}^6_{25}$ lying over $\overline{\mathcal{M}_{23}}$ rather than on the moduli spaces of curves themselves $\overline{\mathcal{M}_{22}}$, $\overline{\mathcal{M}_{23}}$. The map might be ramified and have positive dimensional fibers ands theeefore fail to preserve the conditions on the a, b_i.

The first piece has been taken care of in [LOTZ2] and [JP2]. The first paper uses limit linear series, the second tropical methods. Both are combinatorially very involved

The second piece is still missing.

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