Tropicalizing moduli spaces of algebraic curves + spin curves

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Overview about results tropicalizing moduli spaces of curves + spin curves.

All new results j/w MELO and PACINI (in progress).

set-up: curves$/k=\bar{k}$, char $k \neq 2$.

X projective reduced curve, of genus g.

X smooth: $\mathrm{Pic}(X)$ (= l.b.'s up to $\simeq$ for us)

$$\cup$$

$$\{L : L^2 = \omega_X\} =: S_X \quad \text{S is for "spin".}$$

Note degree of elts of S_X ~~is~~ is $g-1$.

$|S_X| = 2^{2g}$, and S_X splits: $S_X = S_X^- \sqcup S_X^+$

$$S_X^- = \{L : h^0(L) \text{ is odd}\}.$$

$g=3$, X not hyperelliptic: $X \hookrightarrow \mathbb{P}^2$ canonical model as quintic.

odd theta characteristic $\longleftrightarrow$ bitangent line.

I.e. $S_X^- =$ 28 bitangents of canonical model.

Moreover (theorem) 28 lines determine canonical model:

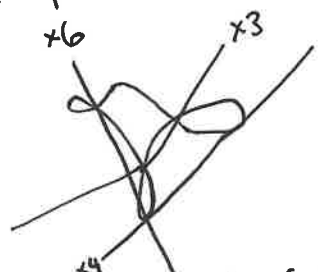
classical flavor, proved ~2000 | Sernesi + Caparaso; Lehrer

{ If 2 plane quintics have the same bitangents, they are equal

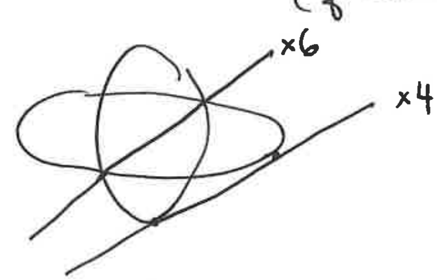
Question: in $((\mathbb{P}^2)^*)^{28}$, can we characterize pts arising as bitangents?

A: yes, used in other proof by Lehrer.

<u>Proof</u>: (Caparaso - Sernesi) Prove by showing for special singular <u>curves</u>. (quintics)



$$4 + 6\cdot 2 + 3\cdot 2^2 = 28$$



$$4 + 6\cdot 2^2 = 28.$$

<u>Fact:</u> type of configuration determines the topological type of the quintic.
+ degeneration argument. This gives results for generic quintic.
Need GIT (nontrivial) to avoid bad curves.

<u>Moduli spaces of curves</u>

$$\begin{tikzcd}
\mathcal{M}_g \arrow[d, hook] & \mathcal{S}_g = \mathcal{S}_g^- \amalg \mathcal{S}_g^+ \arrow[l, "2^{2g}\text{ étale cover}"'] \arrow[d, hook] \\
\overline{\mathcal{M}}_g & \overline{\mathcal{S}}_g \arrow[l]
\end{tikzcd}$$

moduli stack of smooth genus g curves $\to \mathcal{M}_g$

moduli of spin curves $\to \mathcal{S}_g$

$\overline{\mathcal{S}}_g$: compactification by stable spin curves (Disjoint union of compactifications $\overline{\mathcal{S}_g^-}$, $\overline{\mathcal{S}_g^+}$.)

$\overline{\mathcal{M}}_g$: DM compactification

Deligne-Mumford compactification adds certain connected, nodal curves

Want to describe stratifications of compactifications $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{S}}_g$.

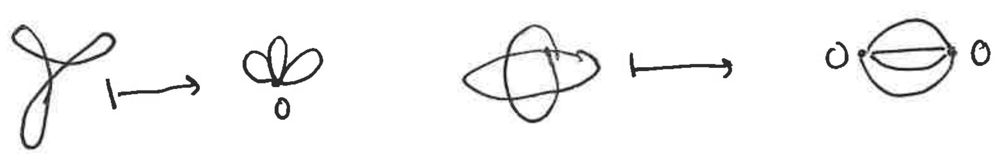
$$\overline{\mathcal{M}}_g \to SG_g \ni G: \ G \text{ is a stable graph } (V, E, w: V \longrightarrow \mathbb{Z}_{\geq 0})$$

stable graphs: SG_g

where the weight function w satisfies $w(v) = 0 \Rightarrow \deg(v) \geq 3$.

note: Graphs are connected

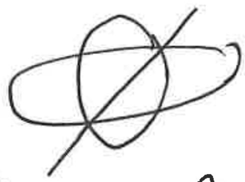
This map: $X \longmapsto G_X = \Big(V := \text{irred. components of } X, \ E = \text{incidence}, \ w: V \to \mathbb{Z}, \ v \mapsto p_g(C_v) \Big)$ (genus of component)

E.g.: 

Caporaso-Jarvis.

<u>Spin curves</u>: stabilizing.

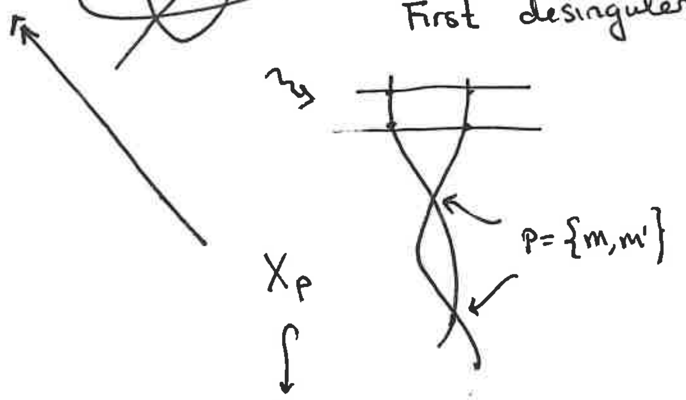
$X \in \overline{M}_g$. recall $S_X = \{(X, L) : L \in Pic(X), L^2 = \omega_X\}$ for smooth X.



X

does not cut a Cartier divisor $\Rightarrow$ no line bundle.

First desingularize (partially)



X_P

$\downarrow$

$\hat{X}_P$

So for general $X \in \overline{M}_g$, $S_X := \{(X_P, L) : L \in Pic(X), L^2 = \omega_X\}$

ranging over $\{\forall P \subset E(G_X) : P \in \mathcal{C}(G_X)\}$

i.e. the set of nodes <u>not</u> resolved

This is the combinatorial version of spin curves for stable curves.

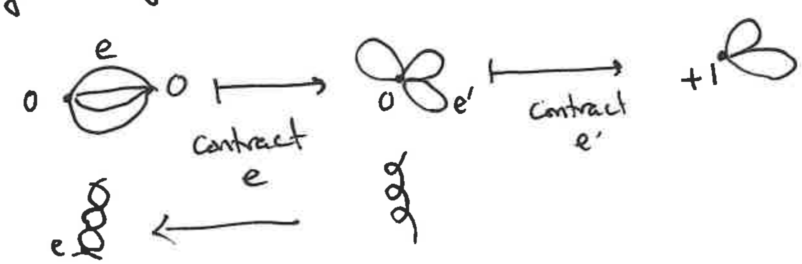
Define $SP_g = \{(G, P, s) : G \in SG_g, P \in \mathcal{C}(G), s \text{ a sign function}\}$.

Upshot: get

$$\begin{array}{ccccc} S_g & \longrightarrow & \overline{S}_g & \twoheadrightarrow & SP_g \\ \downarrow & & \downarrow & & \downarrow \\ M_g & \longrightarrow & \overline{M}_g & \twoheadrightarrow & SG_g \end{array}$$

note! these are <u>posets</u>, which we'll need for tropicalization.

SG_g, SP_g are posets under edge contraction.



when you contract something that's a loop, add weight (genus must not drop!)

The poset structures govern the combinatorial stratifications of

$\overline{\mathcal{M}}_g$ and $\overline{S}_g$.

We actually need to consider marked points (heuristically, because proofs use recursion.)

Have analogous diagram

$$\begin{tikzcd} S_{g,n} \arrow[r, hook] \arrow[d] & \overline{S}_{g,n} \arrow[r, "\sigma"] \arrow[d, "\pi"] & SP_{g,n} \arrow[d] \\ \mathcal{M}_{g,n} \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \arrow[r, "\sigma"] & SG_{g,n} \end{tikzcd}$$

(spin structures ignore markings.)

Example of why worthwhile: $G \in SG_g$.

/exploration/ $\sigma^{-1}(G) = \mathcal{M}_g \longleftarrow \prod_{v \in V} \mathcal{M}_{\omega(v), n_v}$

quotient by automorphism group of graph

marked points must be treated for recursion!

$\Rightarrow$ is irreducible.

Similar statement holds for spin curves:

$S_{(G,P,s)} = \sigma^{-1}((G,P,s))$ has similar (but harder to prove) decomposition. And (theorem) is also irreducible.

(Stable) tropical curves

A (stable) tropical curve is a pair (G, ℓ) where $G \in SG_{g,n}$ and $\ell : E(G) \longrightarrow \mathbb{R}_{>0} \cup \{\infty\}$.

The genus of a tropical curve (G, ℓ) is the genus of G.

A spin tropical curve $\Psi = (G, P, s, \ell)$ where ℓ is as before and $(G, P, s) \in SP_{g,n}$.

These have a moduli space. As sets: it's clear we should have

$$\begin{array}{ccc} \overline{M}^{trop}_{g,n} & \xrightarrow{\tau} & SG_{g,n} \\ \uparrow & & \uparrow \\ \overline{S}^{trop}_{g,n} & \xrightarrow{\tau} & SP_{g,n} \end{array}$$

[note: $\overline{M}^{trop}_g$ not M^{trop}_g because we've allowed ℓ to take value ∞.]

Construction:
We have a ~~sequence~~ collection of cones indexed by a poset + construct spaces as colimit.

Just understand topology of fibers:

$$\tau^{-1}(G) \simeq \frac{(\mathbb{R}_{>0} \cup \{\infty\})^{E(G)}}{\mathrm{Aut}(G)}$$

← really a cone complex!

$$\tau^{-1}(G, P, s) \simeq \frac{(\mathbb{R}_{>0} \cup \{\infty\})^{E(G)}}{\mathrm{Aut}(G, P, s)}$$

Question: can we relate tropical curves/spin curves to algebraic ones in geom. way?

A: "a tropical curve corresponds to an alg. curve over Spec(complete valuation ring)"

Tropicalization: $\pi : \overline{S}_{g,n} \longrightarrow \overline{M}_{g,n}$ apply Berkovich analytification of

thus: $\overline{S}^{an}_{g,n} \xrightarrow{\pi^{an}} \overline{M}^{an}_{g,n}$.

What are these spaces?

A point $\xi \in \overline{\mathcal{S}}_{g,n}^{an}$ over a complete valued field K w/ ring of integers R and valuation $\nu_K : K \longrightarrow \mathbb{R} \cup \{\infty\}$

is precisely an algebraic spin curve:

$$\begin{tikzcd} \xi \arrow[r, leftrightarrow] & \hat{\mathcal{X}} \arrow[r] & \mathrm{Spec}\, R \\ \hat{\mathcal{L}} \arrow[ur, "\text{family of curves}"] & & \\ & & \mathrm{Spec}\, k \arrow[uu, hook] \\ & (\hat{L}, \hat{X}_p) \arrow[ur] & \end{tikzcd}$$

$$\overline{\mathcal{S}}_{g,n}^{an} \xrightarrow{\text{trop}} \overline{\mathcal{S}}_{g,n}^{trop}$$
$$\xi \longmapsto (G_\xi, P_\xi, s_\xi, \ell_\xi)$$

where (G, P, s) is the dual graph of the closed fiber $= \sigma(\hat{L}, \hat{X})$, σ as before $\overline{\mathcal{S}}_{g,n} \longrightarrow SP_{g,n}$

and $\ell_\xi(e) = \nu_K(f_e)$ where f_e is the local equation of e at nodes over e.

(This tells you local geometry over nodes.)

In conclusion:

$$\begin{tikzcd} \overline{\mathcal{S}}_{g,n}^{an} \arrow[r, "\text{trop}"] \arrow[d, "\pi^{an}"'] \arrow[dr, phantom, "\text{(commutes)}"] & \overline{\mathcal{S}}_{g,n}^{trop} \arrow[d, "\text{trop}"] & \simeq \overline{\Sigma}(\overline{\mathcal{S}}_{g,n}^{an}) \\ \overline{\mathcal{M}}_{g,n}^{an} \arrow[r, "\text{trop}", "\text{Abramovich-Payne}"'] & \overline{\mathcal{M}}_{g,n}^{trop} & \simeq \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}^{an}) \end{tikzcd}$$

* Behind this is an isom. of the cone complexes with the skeletons of the corresponding algebraic spaces. *