DESCENDING INVERTIBILITY AND THE BRAUER GROUP OF TOPOLOGICAL MODULAR FORMS

VESNA STOJANOSKA
SPEAKING AT CONNECTIONS FOR WOMEN MSRI
1/28/2018

§1. Preliminaries

Any new results discussed are joint with Ben Antieau and Lennart Meier. Much is not new and is based on work of Gepner, Lawson, and Mathew. In this talk, the goal is to avoid getting too technical – we'll start slowly and get into technical things in a non-technical way.

Invertibility:

- Starting simple, consider a commutative ring R. $x \in R$ is invertible if there exists $y \in R$ such that $xy = 1$. We denote the inverible elements by R^x.
  - If R is a field, then R^x is all nonzero elements.
  - If $R = \mathbb{Z}$, then $R^x = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$.
- Next, consider R-modules: $x \in \mathrm{Mod}_R$ to be invertible if there is a $y \in \mathrm{Mod}_R$ such that $x \otimes y \simeq 1_{\mathrm{Mod}_R} = R$.
  - If R is a field, then the invertible modules are precisely those isomorphic to R (dimension one vector spaces).
  - If $R = \mathbb{Z}$, then there are no nontrivial $\mathbb{Z}$-modules.
- Let's move "one categorical level up": take $x \in \mathrm{Mod}_{\mathrm{Mod}_R}$. What does this mean? The set-up requires some nontrivial technology in higher category theory/ derived algebra. We want to think of $\mathrm{Mod}_{\mathrm{Mod}_R}$ as a category of R-linear categories (hom objects have an R-linear structure). Define an invertible object $x \in \mathrm{Mod}_{\mathrm{Mod}_R}$, to be category such that "$x \otimes y$" $\simeq 1 = \mathrm{Mod}_R$.

  Considering the invertible elements here, we get what we'll call the Brauer group $\mathrm{Br}(B)$.

Even in the most recent definitions, we need to "restrict" the category $\mathrm{Mod}_{\mathrm{Mod}_R}$ to impose some "smallness" constraints. But before we talk more about Brauer groups, we need to enlarge the world of rings.

§2. Rings in homotopy theory

In homotopy theory, we can consider ordinary rings (as discrete ring objects in an appropriate sense), and might want to think about topological rings on occasion But the main "rings" are multiplicative generalized cohomology theories (ring objects in a suitable ∞-category). For example:

- $H^*(-, R)$, ordinary cohomology with coefficients in a ring R.
- $KU^*(-)$, complex topological K-theory.

- $KO^*(-)$, real topological K-theory.
- $TMF^*(-)$, a cohomology theory which is related to elliptic curves and topological modular forms.

These rings live in the category $\mathcal{S}p$ of spectra. This is a symmetric monoidal stable ∞-category, with monoidal unit S^0, the sphere spectrum, which is built from the topological spheres $S^0, S^1, S^2, \ldots$. A few notes:

- In the category of spectra, we can take homotopy groups. $\pi_k X = [S^k, X]$, which form a graded abelian group. (If $k < 0$, we define $\pi_k X = [S^0, S^k \otimes X]$.
- If R is a commutative ring, then $\pi_* X$ form a a graded commutative ring.
- We can build spectra as sequences of spaces, and have the loops infinity/ suspension infinity adjuction,

$$\Sigma^\infty\colon \text{Spaces}_* \leftrightarrow \mathcal{S}p\colon \Omega^\infty.$$

Σ^∞ gives a sequential spectrum by tensoring with S^1.

Now we want to consider "invertible things" in a commutative ring spectrum R. $\Omega^\infty R$ is a space (or more precisely homotopy type), with a multiplicative structure, and $\Omega^\infty R \to \pi_0 R$. Since $\pi_0 R$ is a ring, we can consider its invertible elements. Looking at the fiber over this subset, we obtain the subset $R^x \subset \Omega^\infty R$ of invertible components.

In fact, $R^x = \Omega^\infty \tilde{R}$ for some spectrum $\tilde{R}$, so we can talk about a spectrum of units as well.

Note that $\pi_0 R^x = (\pi_0 R)^x$, and $\pi_t R^x = \pi_t R$ for $t > 0$, so we have a good handle on the homotopy groups of the spectrum of invertible elements in terms of those of R.

§3. Pic in homotopy theory

Mod_R is a symmetric monoidal ∞-category if R is a commutative ring spectrum.

3.1. Example. If R is discrete, the $\text{ho}(\text{Mod}_R)$ is the derived category $\mathcal{D}(R)$.

3.2. Definition. Mimicking previous definitions, let $\underline{Pic}(R)$ denote the set of $x \in \text{Mod}_R$ such that there exists $y \in \text{Mod}_R$ with $x \otimes y \simeq 1_{\text{Mod}_R}$. Because Mod_R is an infinity category, $\underline{Pic}(R)$ is in fact a *space* of invertible R-modules.

Categorically, this is the sub ∞-groupoid (of the category of R-modules) spanned by invertible objects. What kind of space is this?

- $\pi_0 \underline{Pic}(R) = Pic(R)$.
- $\underline{Pic}$ is not just a space but a grouplike E_∞-space. That is, $\underline{P}ic(R)$ has a group structure, and is in the image of $\Omega^\infty(-)$.
- Taking loops, consider $\Omega_{1_{\text{Mod}_R}} \underline{Pic}(R) = R^x$, the spectrum of units discussed before. This tells us about the homotopy groups of the Picard space.

§4. Brauer groups in homotopy theory

The original definition of the Brauer group (due to Brauer?) only applies to fields. For k a field, $\mathrm{Br}(k)$ is defined as the set of central simple algebras over k, modulo Morita equivalence. This has a group structure induced by the tenor product of algebras. For example:

- $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$.
- $\mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ (via the Hasse invariant).

But this definition doesn't even translate to discrete rings. A version for rings was developed by Auslander–Goldman. For homotopy theory (spectra), work has been done by Baker–Richter–Szymik, Antieau–Gepner (in the ∞-categorical setting), Antieau-Lawson, Toën, and others. The upshot is that there's a wealth of new technology to do with Brauer groups.

For the set up: we need some sophisticated technology to deal with "derived" things. The substitute for central algebras are Azumaya algebras.

4.1. Definition. An associative R-algebra A is an **Azumaya algebra** if Mod_A is an invertible object in $\mathrm{Mod}_{\mathrm{Mod}_R}$ (the categeory of R-linear categories vaguely discussed before).

If R is a spectrum, this is equivalent to all of the following three conditions being satisfied:

- A is dualizable as an R-module.
- $A \otimes A^{\mathrm{op}} \simeq \mathrm{End}(A)$.
- The functor $A \otimes (-)$ is conservative.

As R becomes constrained (e.g. R connective, meaning no negative homotopy groups; or R discrete), the definitions recover more familiar notions. This definition is not in the literature – usually one does not work with $\mathrm{Mod}_{\mathrm{Mod}_R}$. One should look at something compactly generated or some other smallness constraint.

4.2. Remark. **Question from audience**: does this definition recover the classical thing for an ordinary ring?
Answer: You'll get (some sort of) derived Brauer group. There are additional "derived" Azumaya algebras that appear.

4.3. Definition. Let $\underline{Az}(R) \subset \underline{Pic}(\mathrm{Mod}_{\mathrm{Mod}_R}) =: Br^{\mathrm{Big}}(R)$ be the connected components containing modules over Azumaya algebras.

Then $\Omega_{\mathrm{Mod}_R}(\underline{Az}(R)) \simeq \underline{Pic}(R)$, so this is a further delooping of $\underline{Pic}(R)$ and we have a way to compute homotopy groups from those of R.

§5. Why would you care?

Answer 1: descent. If we consider the classical versions, descent fails for Picard/ Brauer groups.

5.1. Claim. *$\underline{Pic}(-)$, $\underline{Br}(-)$ satsify étale and Galois descent. (Due to Antieau–Gepner, Gepner–Lawson).*

5.2. Remark. In homotopy theory, Galois extensions are not necessarily étale. In fact, the interesting Galois ones are not.

5.3. Example. $TMF = \Gamma(\mathcal{M}_{\text{ell}}, \mathcal{O}^{\text{top}})$. We can't say much about Brauer groups, but we can compute the locally étale Azumaya algebras:

$$L^{et}Az(TMF) \simeq \mathbb{Z}/24 \hookrightarrow L^{et}Az(\mathcal{M}_{\text{ell}}, \mathcal{O}^{\text{top}}) \twoheadrightarrow \mathbb{Z}/2^{\oplus\infty},$$

where the last sum is countably infinite

§6. Reference suggestions, in response to audience questions

A good place for some definitions is an algebro-geometric paper by Antieau–Gepner, "Brauer groups and étale cohomology in derived algebraic geometry." https://arxiv.org/abs/1210.0290

For learning about spectra, the speaker suggested a historical approach. The classical place to start is Adams' Blue Book ("Stable homotopy and generlised homology," in particular Part III).

Mike Hill is working on a book about how to use derived AG (he spoke up to say this is forthcoming),