Birational geometry of varieties of maximal Albanese dimension

Rita Pardini

Università di Pisa

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A complex torus of dimension q is a quotient $T := V/\Lambda$, where:

- V is a q-dimensional $\mathbb{C}$-vector space
- $\Lambda \subset V$ is a lattice, namely $\Lambda \cong \mathbb{Z}^{2q}$ and $< \Lambda >_{\mathbb{R}} = V$.

The quotient map $p\colon V \to T$ is the universal cover, so V is a complex manifold and $\pi_1(V) = \Lambda$.

T is an abelian variety if there is an embedding $T \hookrightarrow \mathbb{P}^N$.

Riemann's bilinear relations

$T = V/\Lambda$ is an abelian variety iff there exists a positive definite Hermitian form H on V such that $\mathrm{Im} H(\Lambda, \Lambda) \subseteq \mathbb{Z}$.

H as above is a polarization. If $q \geq 2$, not every complex torus T has a polarization.

A smooth complex projective variety X is <u>irregular</u> if $H^0(X, \Omega^1_X) \neq 0$; $q(X) := h^0(X, \Omega^1_X)$ is the <u>irregularity</u>. Being irregular is a topological property: $b_1(X) = 2q(X)$.

If $T = V/\Lambda$ is a complex torus, then for any $\psi \in V^\vee$ the 1-form $d\psi$ descends to a global holomorphic form;
$V^\vee \to H^0(T, \Omega^1_T)$ is an isomorphism, so X is irregular and $q(X) := q$.

More examples: curves of genus $g > 0$, complete intersections in abelian varieties, $X \times Y$ with X irregular, varieties that dominate an irregular variety...

In particular, if $f\colon X \to T = V/\Lambda$ is nonconstant, then X is irregular.
Let $\omega_i = f^* dz_i$, where $z_1, \dots z_q$ are coordinates on V; locally near any $x_0 \in X$ the map f lifts to $V \cong \mathbb{C}^q$ as

$$x \mapsto \left(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_q \right) + c$$

for some $c \in V$.

To get a similar global expression, we need to mod out the periods, i.e., the integrals $(\int_\gamma \omega_1, \dots, \int_\gamma \omega_q)$ for $\gamma \in H_1(X, \mathbb{Z})$.

Warning: if $\omega_1, \dots \omega_k$ are holomorphic 1-forms, in general the periods $(\int_\gamma \omega_1, \dots, \int_\gamma \omega_k)$ do **not** form a lattice in $\mathbb{C}^k$.

Hodge theory $\Rightarrow$ get a lattice if one takes **all** the holomorphic 1-forms:

- set $V := H^0(X, \Omega^1_X)^\vee$
- the image $\Lambda \subset V$ of the map $H_1(X, \mathbb{Z}) \to V$ defined by $\gamma \mapsto \int_\gamma -$ is a lattice.

So we have the Albanese torus $\mathrm{Alb}(X) := V/\Lambda$ and the Albanese map $a_X \colon X \to \mathrm{Alb}(X)$, $x \mapsto \int_{x_0}^{x} -$, where $x_0 \in X$ is a base point.

$\mathrm{Alb}(X)$ is actually an **abelian variety** (the polarization is induced by a choice of an ample line bundle on X).

Universal property: given $f \colon X \to T$ holomorphic map, f factorizes uniquely as $X \overset{a_X}{\to} \mathrm{Alb}(X) \to T$.

If $f(X)$ generates T as a group then T is an abelian variety.

The Albanese dimension of X is $\text{albdim}(X) := \dim(a_X(X))$;
X has maximal Albanese dimension (m.A.d) if
$\text{albdim}(X) = \dim X =: n$.

This is a topological property: $\text{albdim}(X) \geq k$ iff
$\wedge^{2k} H^1(X, \mathbb{C}) \to H^{2k}(X, \mathbb{C})$ is not the zero map.

Let L be a big line bundle, i.e. s.t. $|L^{\otimes m}|$ gives a generically injective map $X \to \mathbb{P}^N$ map for $m \gg 0$.
A numerical measure of bigness is the volume

$$\text{vol}(L) := \lim_m n! \frac{h^0(X, L^{\otimes m})}{m^n} \in \mathbb{R}_{>0}$$

Note: if L is nef, $\text{vol}(L) = L^n$.

Let $\omega_X = \mathcal{O}_X(K_X)$ canonical sheaf: X is <u>of general type</u> if ω_X is big.
For X of general type we consider two numerical birational invariants:

$$\mathrm{vol}(X) := \mathrm{vol}(\omega_X) \qquad \text{and} \quad \chi(X) := \chi(\omega_X) = \sum_{i=0}^{n} (-1)^i h^i(X, \omega_X).$$

Geographical problem: what are the restrictions on vol(X) and $\chi(X)$? and what if X is irregular/m.A.d.?

If $n = 2$, then $\text{vol}(X), \chi(X) \in \mathbb{N}_{>0}$ and:

- $\text{vol}(X) \leq 9\chi(X)$ (Bogomolov–Miyaoka–Yau inequality)
- $\text{vol}(X) \geq 2\chi(X) - 6$ (Noether inequality)
- $\text{vol}(X) \geq 2\chi(X)$ if X is irregular (Bombieri)
- $\text{vol}(X) \geq 4\chi(X)$ if X is m.A.d. (“Severi inequality”, P. ’04)

If $n > 2$ and X of m.A.d, then:

- $\chi(X) \geq 0$
- $\text{vol}(X) \geq 2n!\chi(X)$ (“Generalized Severi inequality", Barja ’14 and Tong Zhang ’14)

We set $\mathrm{Pic}^0(X) := \{\text{topol. trivial l. bundles on } X\}/\cong$;
if $T = V/\Lambda$, $\mathrm{Pic}^0(T)$ the dual torus, a complex torus of dimension q.
In general, the map $a_X^*\colon \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ is an isomorphism.

Generic vanishing (Green–Lazarsfeld '87): for X m.A.d.,
$H^i(X, \omega_X \otimes \alpha) = 0$ for $i > 0$ and $\alpha \in \mathrm{Pic}^0(X)$ general.

So for $\alpha \in \mathrm{Pic}^0(X)$ general:

- $\chi(X) = \chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(\omega_X \otimes \alpha) \geq 0$
- the generalized Severi inequality can be written $\mathrm{vol}(\omega_X) \geq 2n!h^0(\omega_X \otimes \alpha)$.

Generalized set-up:

$a\colon X \to A$ a map to an abelian variety such that:

- a is generically finite onto its image
- $\mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is injective

We set: $q := \dim A$ and we fix $L \in \mathrm{Pic}(X)$.

The continuous rank of L (with respect to a) is:

$$h^0_a(X, L) := \min\{h^0(X, L \otimes \alpha) \mid \alpha \in \mathrm{Pic}^0(A)\}.$$

(generic vanishing $\Rightarrow h^0_a(\omega_X) = \chi(X)$).

Basic diagram:

Let d be an integer and let $\mu_d\colon A \to A$ be multiplication by d;

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{\mu_d}} & X \\ {\scriptstyle a_d}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{\mu_d} & A \end{array}$$

$X^{(d)}$ is connected,

$\widetilde{\mu_d}$ is a degree d^{2q} étale cover,

the map a_d satisfies the same properties as a.

Set $L^{(d)} := \widetilde{\mu_d}^* L$; we wish to study $|L^{(d)} \otimes \alpha|$ for $\alpha \in \mathrm{Pic}^0(A)$ general and $d \gg 0$.

Multiplicative property of the continuous rank:

$$h^0_{a_d}(X_d, L^{(d)}) = d^{2q} h^0_a(X, L)$$

If $h^0_a(L) > 0$, we define the slope $\lambda(L) := \frac{\mathrm{vol}(L)}{h^0_a(L)}$.

Note: mult. property $\Rightarrow \lambda(L^{(d)}) = \lambda(L)$

Key remark (Barja): if $h^0_a(X, L) > 0$, then $|L^{(d)}|$ gives a generically finite map for $d \gg 0$.
$\Rightarrow L^{(d)}$ is big $\Rightarrow L$ is big and $\lambda(L) > 0$.

A <u>Clifford-Severi inequality</u> is an inequality of the form

$$\lambda(L) \geq C(n),$$

where $C(n) > 0$ is a constant depending on n.

Theorem (Barja '14):

- if L is nef, then $\lambda(L) \geq n!$ (1^{st} C-S inequality)
- if L is nef and $\omega_X \otimes L^{-1}$ is pseff, then $\lambda(L) \geq 2n!$ (2^{nd} C-S inequality)

The generalized Severi inequality is a consequence of Barja's Theorem for $L = \omega_X$.

In recent joint work with Barja and Stoppino we introduced:

- a new type of asymptotic study for line bundles on m.A.d. varieties (“eventual map”)
- the “continuous continuous rank function”.

We have used these to drastically simplify the proofs of the known Clifford-Severi inequalities and improve them.

Theorem (Refined C-S inequalities, Barja–P.–Stoppino '16):

If $h^0_a(L) > 0$, then:

1. $\lambda(L) \geq n!$
2. $\lambda(L) \geq 2n!$ if $\omega_X \otimes L^{-1}$ pseff
3. $\lambda(L) \geq \frac{5}{2} n!$ if $\omega_X \otimes L^{-1}$ pseff, $n \geq 2$ and a gen. inj.
4. $\lambda(L) \geq \frac{9}{4} n!$ if $\omega_X \otimes L^{-1}$ pseff, $n \geq 2$ and a is not composed with an involution.

Remarks:

- (1) and (2) were proven by Barja for L nef.
- All statements are simplified versions (need a version for pairs $T \subseteq X$ for the proof).
- If X is a minimal surface and $L = \omega_X$, then (3) gives $K_X^2 \geq 5\chi(X)$; it is conjectured that $K_X^2 \geq 6\chi(X)$ when a is gen. injective.
- (4) extends a result of Lu-Zuo for $n = 2$ and $L = \omega_X$

The eventual degree

Assume $h^0_a(L) > 0$.

For $d \in \mathbb{N}$, let $m_L(d)$ be the degree of the map given by $|L^{(d)} \otimes \alpha|$ for $\alpha \in \mathrm{Pic}^0(A)$ general.

The eventual degree of L is:

$$m_L := \min\{m_L(d) \mid d \in \mathbb{N}\}.$$

Remark: $m_L < +\infty$ (by Barja's observation) and $m_L(d) = m_L$ for $d \gg 0$.

The eventual map

Theorem (Eventual factorization):

There exists a generically finite dominant map $\varphi_L \colon X \to Z$ of degree m_L (the eventual map) such that:

(a) the map a factorizes as $X \xrightarrow{\varphi_L} Z \to A$

(b) consider the cartesian diagram:

$$\begin{array}{ccccc} X^{(d)} & \xrightarrow{\varphi^{(d)}} & Z^{(d)} & \longrightarrow & A \\ \widetilde{\mu_d}\downarrow & & \downarrow & & \downarrow\mu_d \\ X & \xrightarrow{\varphi_L} & Z & \longrightarrow & A \end{array}$$

then $|L^{(d)} \otimes \alpha|$ is birationally equivalent to $\varphi^{(d)}$ for α general and $d \gg 0$.

Remarks:

- the statement is birational: φ_L is unique up to birational isomorphism
- this is a new way of associating a map with a line bundle
- there is a formal analogy between the eventual map and the Iitaka fibration, in a situation where the Iitaka fibration is birational and gives no information.
- For $L = \omega_X$ and $A = \mathrm{Alb}(X)$, we have the eventual paracanonical map, which is a new geometrical object attached to X.

Corollary:

- If a is birational, then φ_L is birational
- If a is not composed with an involution, then $m_L \neq 2$.

This will be crucial in proving some of the numerical inequalities.

Covering trick I:

(pullbacks from A become divisible)

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{\mu_d}} & X \\ a_d \downarrow & & \downarrow a \\ A & \xrightarrow{\mu_d} & A \end{array}$$

Fix H very ample on A and set $M = a^* H$, $M_d := a_d^* H$.
The line bundle M_d is big and base point free.

On the other hand, $\mu_d^* H \equiv_{alg} d^2 H$, so $\frac{1}{d^2} M^{(d)} \equiv_{\mathrm{Pic}^0(A)} M_d$ is an integral class.

Let $x \in \mathbb{Q}$ and let $d \in \mathbb{N}$ be such that $d^2 x = e \in \mathbb{Z}$. We set:

$$\phi(x) := \frac{1}{d^{2q}} h^0_{a_d}\left(X^{(d)}, (L + xM)^{(d)}\right) = \frac{1}{d^{2q}} h^0_{a_d}(X^{(d)}, L^{(d)} + eM_d)$$

By the multiplicative property of the continuous rank, this is well defined.

Properties: For $x_1 < x_2 \in \mathbb{Q}$ we have:

- $\phi(x_1) \leq \phi(x_2)$
- $2\phi(\frac{x_1+x_2}{2}) \leq \phi(x_1) + \phi(x_2)$ ("midpoint property")

$\implies$ ϕ extends to a convex continuous non decreasing function $\phi \colon \mathbb{R} \to \mathbb{R}$.

So the function ϕ is right and left differentiable at every point and differentiable outside a countable set.

Proposition: for any $x \in \mathbb{R}$:

$$D^{-}\phi(x) := \lim_{d \to +\infty} \frac{1}{d^{2q-2}} h^0_{a_d}(X_{|M_d}, (L + xM)^{(d)}),$$

where $M_d \in |M_d|$ is general and $h^0_a(X_{|M_d}, L + xM)$ is the <u>restricted continuous rank</u>.

Idea of the proofs

Consider the 2^{nd} Clifford–Severi inequality:

$$\omega_X \otimes L^{-1} \text{ pseff } \implies \text{vol}(L) \geq 2n!h^0_a(X, L)$$

The proof is by induction on n.
We start with $n = 1$:

Covering trick, II:

(elimination of lower order terms in inequalities)

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{\mu_d}} & X \\ {\scriptstyle a_d}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{\mu_d} & A \end{array}$$

Since $\deg(\omega_X \otimes L^{-1}) \geq 0$, we apply Clifford's theorem to $L^{(d)}$:

$$d^{2q} \deg L = \deg L^{(d)} \geq 2h^0(L^{(d)}) - 2 \geq d^{2q} h^0_a(L) - 2$$

divide by d^{2q} and let $d \to \infty$:

$$\deg L \geq 2h^0_a(L), \quad \text{i.e., } \lambda(L) \geq 2$$

Inductive step:

Set $\psi(x) := \mathrm{vol}(L + xM)$ and $\bar{x} := \max\{t \mid \mathrm{vol}(L + xM) = 0\}$.

Theorem (Lazarsfeld–Mustaţă, Boucksom–Favre–Jonsson '09):

ψ is differentiable for $x > \bar{x}$ and one has:

$$\psi'(x) = n\,\mathrm{vol}_{X|M}(L + xM)$$

Inductive hypothesis $\implies$

$$\psi'(x) = n\,\mathrm{vol}_{X|M}(L + xM) \geq n \cdot 2(n-1)!\phi'(x) = 2n!\phi'(x)$$

Taking $\int_{-\infty}^{0}$ of both sides of the above equation gives:

$$\mathrm{vol}(L) \geq 2n!h_a^0(L).$$

□

Proof of the refined inequalities (when a is either generically injective or not composed with an involution):

- since $\lambda(L) = \lambda(L^{(d)})$ we may replace X by $X^{(d)}$, for $d \gg 0$, and assume that the map given by $|L|$ is replaced by the eventual map.
- in the first inductive step ($n = 2$) use refined versions of the inequality, that hold when $|L|$ is either birational or not composed with an involution.