MODULI SPACES OF PARABOLIC VECTOR BUNDLES ON $\mathbb{P}^1$

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Connections for Women: Derived Algebraic Geometry
Birational Geometry and Moduli Spaces

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Parabolic vector bundles on $\mathbb{P}^1$

- $p_1, \ldots, p_n \in \mathbb{P}^1$ general $(n \geq 5)$
- $\mathbb{E} = (E, V_i)$
  - E rank 2 vector bundle on $\mathbb{P}^1$, $\deg(E) = 0$
  - $V_i \subset E_{p_i}$ 1-dimensional linear subspace
- $A = (a_1, \ldots, a_n) \in [0, 1]^n$
- $\deg_A$, μ_A, μ_A-(semi)stability
- $\overline{\mathcal{M}}_A$ moduli space of μ_A-semistable $\mathbb{E} = (E, V_i)$

Remark

$\mathcal{M}_A \subset \overline{\mathcal{M}}_A$ Zariski open subset parametrizing μ_A-stable bundles

If $\mathcal{M}_A \neq \emptyset$, then $\mathcal{M}_A$ is smooth of dimension $n - 3$

Parabolic vector bundles on $\mathbb{P}^1$

$(\mathbb{P}^1, p_1, \ldots, p_n)$

$A = (a_1, \ldots, a_n) \in (0,1)^n$

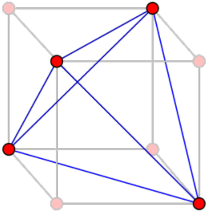
$\overline{\mathcal{M}}_A$ moduli space of μ_A-semistable $\mathbb{E} = (E, V_i)$

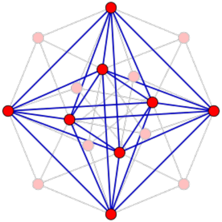
Theorem (Bauer 1991)

- *Weight polytope* $\Delta \subset [0,1]^n$
- *Chamber decomposition on* Δ $(A \equiv A')$
- *Birational maps between different models*

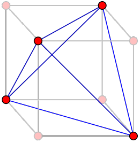
The demihypercube $\Delta \subset [0,1]^n$

$\Delta = \langle v \text{ even vertex of } [0,1]^n \rangle$





The demihypercube $\Delta \subset [0,1]^n$



$$\mathrm{Aut}(\Delta) \;=\; W(D_n) \;\cong\; \big(\mathbb{Z}/2\mathbb{Z}\big)^{n-1} \rtimes S_n$$

Moduli realization of $\big(\mathbb{Z}/2\mathbb{Z}\big)^{n-1}$ via elementary transformations

Moduli realization of $(\mathbb{Z}/2\mathbb{Z})^{n-1} \subset \mathrm{Aut}(\Delta)$

$$\mathbf{El} \;=\; \{\text{ even elementary transformations }\} \;\cong\; \left(\tfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1}$$

$$\mathbf{El}_A \;=\; \left\{\varphi \in \mathbf{El} \;\middle|\; A \equiv A^{\varphi}\;\right\} \;\subset\; \mathrm{Aut}(\overline{\mathcal{M}}_A)$$

Define a subpolytope $\Delta \supset \Sigma \ni A_F = (\frac{1}{2}, \ldots, \frac{1}{2})$

Theorem (A-Fassarella-Kaur-Massarenti 2019)

For $A \in \Sigma$

$$\mathrm{Aut}(\overline{\mathcal{M}}_A) \;=\; \mathbf{El}_A$$

Mori Dream Space (MDS)

Definition (Hu-Keel 2000)

- X $\mathbb{Q}$-factorial projective variety such that $\mathrm{Pic}(X)_{\mathbb{Q}} = N_1(X)$
- Cox ring

$$Cox(X) = \sum_{D \in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D))$$

- X is a MDS if $Cox(X)$ is finitely generated

Example

Toric varieties (Cox 1995)

The birational geometry of X can be encoded in its cone of effective divisors $\mathrm{Eff}(X)$ together with a finite chamber decomposition on it

Mori Dream Space (MDS)

Example

$$X \; = \; Bl_{p_1,p_2}\mathbb{P}^4 \;\xrightarrow{\pi}\; \mathbb{P}^4$$

- $H = \pi^*\mathcal{O}_{\mathbb{P}^4}(1)$
- E_i exceptional divisor over p_i
- H_i strict transform of hyperplane through p_i
- H_{12} strict transform of hyperplane through p_1 and p_2

- $\mathrm{Mov}(X) \; = \; \mathrm{Nef}(X) \; \cup \; \mathrm{Nef}(X^+)$

- $-K_X \in \mathrm{Int}\big(\mathrm{Nef}(X^+)\big) \quad \Longrightarrow \quad X^+$ is Fano

Mori Dream Space (MDS)

Theorem (Hu-Keel 2000)

X Mori Dream Space admits a Mori chamber decomposition

$$\mathrm{Mov}(X) = \mathrm{Nef}(X) \cup \mathrm{Nef}(X_1) \cup \cdots \cup \mathrm{Nef}(X_k)\ , \quad X_i \cong_{pseudo} X$$

Theorem (Mukai 2001, Castravet-Tevelev 2005)

For $n \geq 5$

$$X = Bl_{k\ pts}\mathbb{P}^n \text{ is a MDS} \iff k \leq n+3$$

$X = Bl_{n\ \mathrm{pts}}\mathbb{P}^{n-3}$ general

$X = Bl_{n \text{ pts}} \mathbb{P}^{n-3}$

THEOREM (MUKAI 2005 , A.-MASSARENTI 2015)

$\exists$ *linear projection*

$$\pi : \mathbb{R}^{n+1} \dashrightarrow \mathbb{R}^n \quad \textit{explicit}$$

- $\pi(\mathrm{Eff}(X)) = \Delta$
- $\pi(\mathrm{Mov}(X)) = \Sigma \subset \Delta$
- $\mathrm{Mov}(X) = \mathrm{Nef}(X) \cup \mathrm{Nef}(X_1) \cup \cdots \cup \mathrm{Nef}(X_k)$
  $\leftrightarrow$ *weight chamber decomposition on* Σ
- $\pi(-K_X) = A_F = (\frac{1}{2}, \ldots, \frac{1}{2}) \in \Sigma$

$\overline{\mathcal{M}}_{A_F}$ is a Fano variety - smooth if n is odd

The Fano variety $\overline{\mathcal{M}}_{A_F}$ (of even dimension $n-3$)

Example ($n = 5$)

$X \ = \ Bl_{5\text{ pts}}\mathbb{P}^2$ general

- $X \ \cong \ Q_1 \cap Q_2 \ \subset \ \mathbb{P}^4$
- $\mathrm{Eff}(X) \ = \ \langle 16 \text{ lines} \rangle$
- $\ell \ \rightsquigarrow \ \ell_1, \cdots, \ell_5 \ \rightsquigarrow \quad \begin{aligned} \pi_\ell \ : X \ &\to \ \mathbb{P}^2 \\ \ell_i \ &\mapsto \ p_i \end{aligned}$

The Fano variety $\overline{\mathcal{M}}_{A_F}$ (of even dimension $n-3$)

$X \ = \ Bl_{n \text{ pts}}\mathbb{P}^{n-3} \ , \quad n-3 = 2m \geq 4$

- $Z \ = \ Q_1 \cap Q_2 \ \subset \ \mathbb{P}^{2m+2}$ general
- $G \ = \ \{L = \mathbb{P}^{m-1} \mid L \subset Z\} \ \subset \ \mathbb{G}(m-1, \mathbb{P}^{2m+2})$
- G is a Fano variety , $\dim(G) = 2m$

Theorem (Casagrande 2015)

$$G \ \cong_{pseudo} \ X$$

The Fano variety $\overline{\mathcal{M}}_{A_F}$ (of even dimension $n-3$)

$Z \; = \; Q_1 \cap Q_2 \; \subset \; \mathbb{P}^{2m+2} \; , \quad n-3 = 2m$ general

- $Q_1 : \sum_{i=1}^{n} \; x_i^2 = 0$ and $Q_2 : \sum_{i=1}^{n} \; \lambda_i x_i^2 = 0$
- $\mathcal{M} \; = \; \{ M = \mathbb{P}^m \mid M \subset Z \} \qquad \# \mathcal{M} = 2^{2m+2}$
- $\sigma_i \; : Z \; \to \; Z$, $\quad x_i \; \mapsto \; -x_i \qquad \mathrm{Aut}(Z) \; = \; \langle \sigma_i \rangle \; \cong \; \left(\mathbb{Z}/2\mathbb{Z} \right)^{2m+2}$

$G \; = \; \{ L = \mathbb{P}^{m-1} \mid L \subset Z \}$

- $M \in \mathcal{M} \quad \rightsquigarrow \quad E_M = \; \{ L = \mathbb{P}^{m-1} \mid L \cap M \neq \emptyset \} \in \; \mathrm{Eff}(G)$

The Fano variety $\overline{\mathcal{M}}_{A_F}$ (of even dimension $n-3$)

$X = Bl_{n \text{ pts}}\mathbb{P}^{n-3}$, $n-3 = 2m \geq 4$

$Z = Q_1 \cap Q_2 \subset \mathbb{P}^{2m+2}$

$G = \{ L = \mathbb{P}^{m-1} \mid L \subset Z \}$

Theorem (A.-Casagrande 2017)

- $\mathrm{Eff}(G) = \langle E_M \rangle_{M \in \mathcal{M}}$, $\#\mathcal{M} = 2^{2m+2}$
- $M \in \mathcal{M} \rightsquigarrow M_1 = \sigma_1(M), \cdots, M_n = \sigma_n(M) \rightsquigarrow \exists\, !$

$$
\begin{array}{ccccc}
G & \overset{\cong_{pseudo}}{\dashrightarrow} & X & \to & \mathbb{P}^{n-3} \\
E_{M_i} & \dashrightarrow & E_i & \mapsto & p_i \\
E_M & \dashrightarrow & & & Sec_{m-1} C_{n-3}
\end{array}
$$

Proof

Step 1

$X \;=\; Bl_{n \text{ pts}}\mathbb{P}^{n-3}\;, \qquad n-3=2m$

Explicit

$$\mathrm{Mov}(X) \;=\; \mathrm{Nef}(X) \;\cup\; \mathrm{Nef}(X_1) \;\cup\; \cdots \;\cup\; \mathrm{Nef}(G) \;\cup\; \cdots \;\cup\; \mathrm{Nef}(X_k)$$

$\rightsquigarrow$ Description of

$$\mathrm{Nef}(G) \subset \mathrm{Mov}(G) \subset \mathrm{Eff}(G)$$

in terms of generators of

$$\mathrm{Pic}(X) \;=\; \langle\; H, E_1, \cdots, E_n \;\rangle$$

Proof

Step 2

Description of

$$\mathrm{Nef}(G) \subset \mathrm{Mov}(G) \subset \mathrm{Eff}(G)$$

in terms of $\{E_M\}_{M \in \mathcal{M}} \subset \mathrm{Pic}(G)$

$\mathcal{I} := \{(L, p) \in G \times Z \mid p \in L\}$

$$\begin{array}{ccc} & \mathcal{I} & \\ \swarrow & & \searrow \\ G & & Z \end{array}$$

$$\begin{array}{ccccc} H_2(G, \mathbb{Z}) & \overset{\cong}{\to} & H^{2m}(Z, \mathbb{Z}) & \overset{\cong}{\to} & H^2(G, \mathbb{Z}) \\ \ell \subset M^* & \mapsto & M & \mapsto & E_M \end{array}$$

Proof

Step 3

$\varphi : G \overset{\cong_{pseudo}}{\dashrightarrow} X$

$\varphi^* : H^2(X, \mathbb{R}) \overset{\cong}{\to} H^2(G, \mathbb{R})$

$\exists M \quad \forall i \quad \varphi^*(E_i) = E_{\sigma_i(M)}$

Act on G with $\langle \sigma_i \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^{2m+2}$

□

Thank you!