Bounding singularities on general type surfaces

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Geography of Surfaces

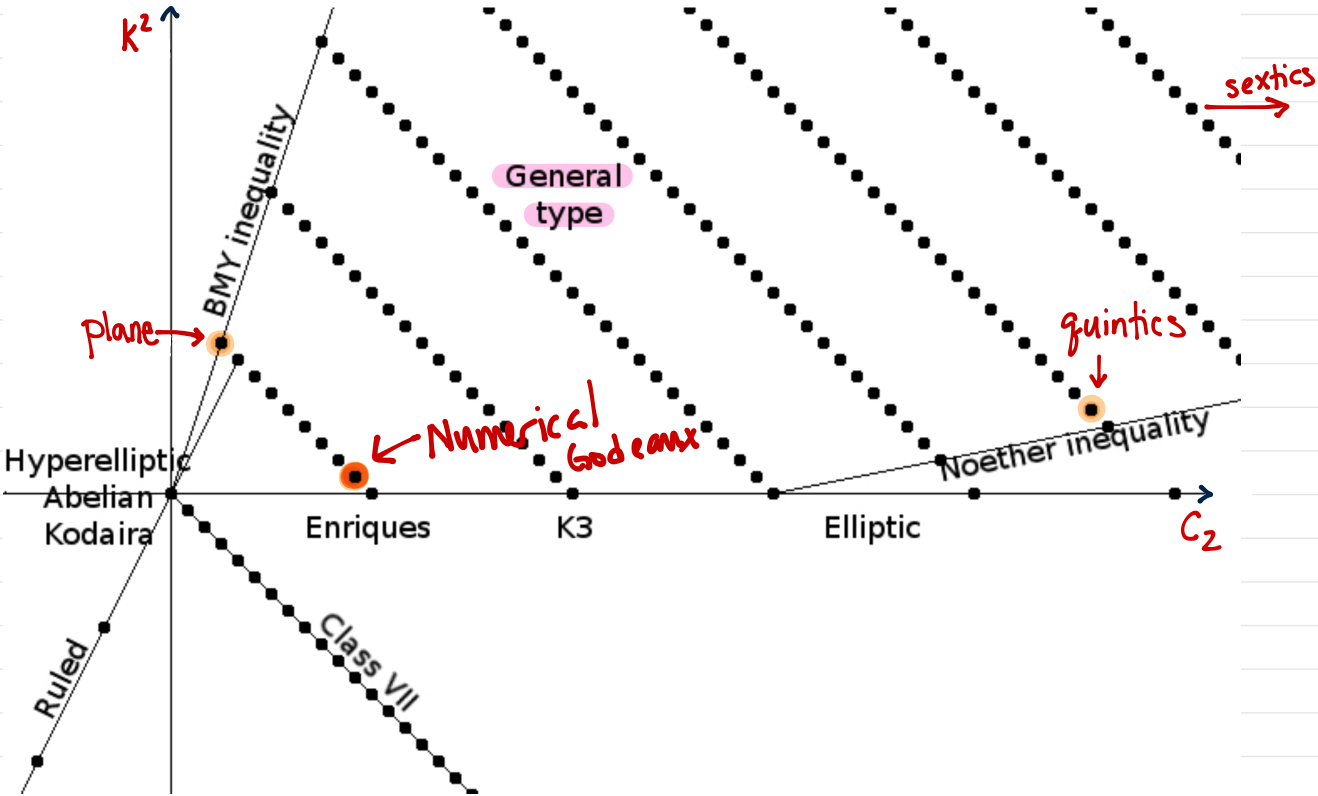
Let S be a smooth, minimal, projective surface over $\mathbb{C}$.

$\kappa(S)$ = Kodaira dimension of S
= maximal dimension of image of $\varphi_{|mK_S|} : S \dashrightarrow \mathbb{P}^N$, $m \in \mathbb{Z}_{>0}$
$= -\infty, 0, 1,$ or 2 ← general type

invariants: K_S^2 = square of the canonical class
χ = holomorphic Euler characteristic

ex: quintic in $\mathbb{P}^3$: $K^2 = 5$, $\chi = 5$

Geography of Surfaces



(Source: Wikipedia)

The playground

$\mathcal{M}_{K^2,\chi}$ = moduli space of smooth surfaces of general type with fixed invariants

$\overline{\mathcal{M}}^{KSBA}_{K^2,\chi}$ = KSBA compactification of stable surfaces of general type
(slc singularities, $\mathbb{Q}$-Gorenstein deformations)

The dream: Understand birational geometry of $\overline{\mathcal{M}}^{KSBA}_{K^2,\chi}$

ex: $\varphi_{|L|} : \overline{\mathcal{M}} \dashrightarrow$???

A key ingredient: Divisors (corresponding to singular surfaces)

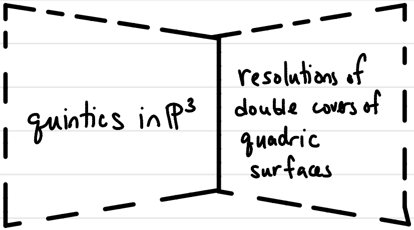
Interested in playing?

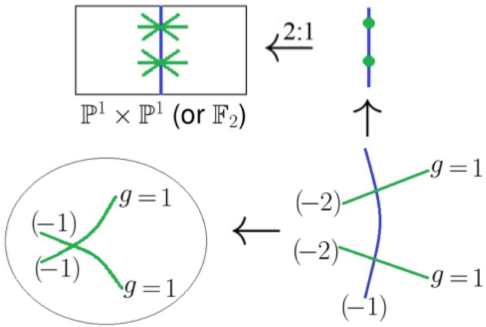
Key tools:
* birational geometry
* deformation theory
* (a bit of) combinatorics
* singularity theory
* (a bit of) representation theory
* with nifty connections to symplectic geometry!

Where can we find a divisor?

ex: $\overline{\mathcal{M}}_{5,5}$ = moduli space of stable (numerical) quintic surfaces

(Horikawa) $\mathcal{M}_{5,5}$





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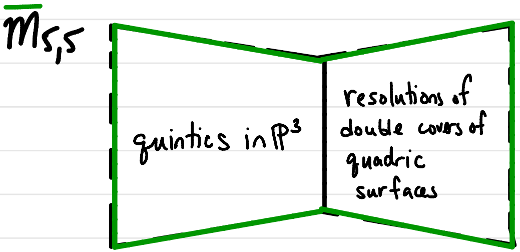


Theorem. (– '17) Stable quintic surfaces with a unique $\frac{1}{4}(1,1)$ singularity form a divisor in $\overline{M}_{5,5}$.

$\frac{1}{4}(1,1)$ singularity: $\mathbb{C}^2/\mu_4$ with action $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$

Q: How to generalize?

Cyclic Quotient Singularities

$$\tfrac{1}{n}(1,a) = \mathbb{C}^2/\mu_n \quad \text{via} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^a \end{pmatrix}$$

$\uparrow$ minimal resolution

where $\displaystyle \frac{n}{a} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$

Q: Which are smoothable?

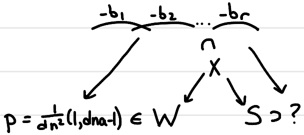
A: (locally) $\frac{1}{dn^2}(1, dna-1)$

Moral: Surfaces with a unique $\frac{1}{n^2}(1, na-1)$ singularity should form a divisor in $\overline{\mathcal{M}}_{K^2,\chi}$!

Q: Can we please narrow the search?

Q': Boundedness results (Alexeev) $\Rightarrow$ n is bounded.
Can we make this bound sharper? Enough to bound $r-d$!

The result



Theorem (—, Urzúa). Let $\kappa(S)$ be the Kodaira dimension of S.

1. If $\kappa(S)=0$, then $r-d \leq 4K_W^2$
2. If $\kappa(S)=1$, then $r-d \leq 4K_W^2-2$
3. If $\kappa(S)=2$, then $r-d \leq 4(K_W^2-K_S^2)-4$ when $K_W^2-K_S^2>1$, $r-d \leq 1$ otherwise

In all 3 cases, the bounds are optimal.

4. If $\kappa(S)=-\infty$, then S is rational and $r-d \leq 4(K_W^2-K_S^2)-2K_S \cdot \pi(C)$

Rmk. $\chi(S)=\chi(W)$, so S is "usually" not rational.

Deformations of W are governed by those of S: can be hard to study!!

From here...

Q: Bounds in case of multiple singularities?

Q: Are these divisors ample?

Q: How do boundary divisors intersect?

Q: Intersection with Gorenstein locus?

ex: (Franciosi-Pardini-Röllenske) Classification of Gorenstein stable Godeaux ($K^2 = \chi = 1$)

Q: Non-normal singularities?

Thanks for your attention!