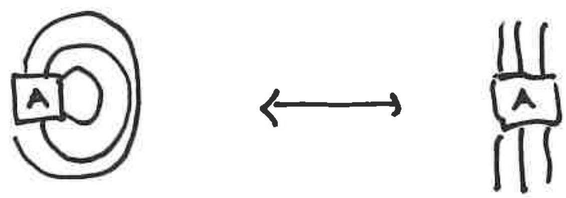
Tina Kanstrup – Link homologies and Hilbert Schemes. 1/30/2019 ①

Research area – geometric representation theory.

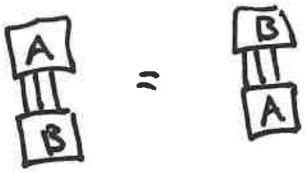
Link – a collection of knots (that may be interlinked).

Theorem (Markov). Links up to isotopy $\longleftrightarrow$ braid groups on n strands up to some equivalence.

$\bigsqcup_n B_n/\sim$



Markov moves: encode relations on $\bigsqcup_n B_n$.

(1) 

(2) 

Khovanov-Rozansky link homology (HHH)

Fix $n \in \mathbb{N}$. $R = \mathbb{Q}[x_1, \dots, x_n] \circlearrowleft S_n$ $(\deg x_i = 2)$. $s = (i\ i{+}1)$ simple reflection.

$B^s := R \otimes_{R^s} R(1)$.

$\mathbb{S}\mathrm{Bim} \subset$ graded R-R bimodules

smallest containing R, B^s and closed under $\oplus$, $\otimes$, ± 1, $\overset{\oplus}{\subset}$

$F_s := [B^s \longrightarrow R(1)]$, $F_{s^{-1}} := [R(-1) \longrightarrow B_s]$

$\beta = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}$. $F(\beta) := \bigotimes_j F_{s_{i_j}^{\varepsilon_j}}$

$HHH^{\bullet,0,\bullet}(B) = H^\bullet(\mathrm{Hom}^\bullet_{\mathbb{S}\mathrm{Bim}}(R, F(\beta)))$

link invariant.

<u>Conjecture</u>. (Gorsky – Neguț – Rasmussen)

$\exists$ a pair of adjoint functors

$$K^b(\mathbb{S}\mathrm{Bim}) \underset{\iota^*}{\overset{\iota_*}{\rightleftarrows}} D^b\mathrm{Coh}^{\mathbb{C}^*\times\mathbb{C}^*}\left(\mathrm{FHilb}_n^{dg}(\mathbb{C}^2,\mathbb{C})\right)$$

such that (1) ι^* is monoidal and fully faithful

(2) $H^*\left(\iota_*(F(\beta))\right) = HHH(\beta)$.

Everything is type A, i.e. $G = Gl_n \supset B = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\}$

$\mathfrak{g} = \mathrm{Mat}_n \supset \mathfrak{b} = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\} \supset \mathfrak{n} = \left\{\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}\right\}$

$$\mathrm{FHilb}_n^{dg}(\mathbb{C}^2,\mathbb{C}) = \left\{(x,y,v) \in \mathfrak{b} \overset{R}{\underset{\mathfrak{n}}{\times}} \mathfrak{n} \times \mathbb{C}^n \mid \mathbb{C}\langle x,y\rangle v = \mathbb{C}^n\right\}$$

<u>Current status</u>: progress due to Oblomkov – Rozansky:

Construct a link invariant in terms of matrix factorizations.

(conjecturally, their invariant is HHH.)

Kanstrup-Bezrukavnikov – Neguț – Okounkov: relate O-R construction to the Gorsky – Neguț – Rasmussen conjecture.

Why derived AG?

<u>Affine picture</u>. G reductive algebraic group.

<u>Thm</u> (Bezrukavnikov – Riche). There is a categorical action of extend affine braid group (which contains ordinary braids) on

$$\underbrace{D^b\mathrm{Coh}^{G\times\mathbb{C}^*}\left(\frac{G\times\mathfrak{n}}{B} \times_{\mathfrak{g}} \frac{G\times\mathfrak{n}}{B}\right)}_{\text{"affine Hecke category"}}.$$

The action is by convolution w/ certain objects

St: Steinberg

Thm (Bezrukavnikov, Bez-Yun)

$$D^b Coh^{G\times\mathbb{C}^*}(St) \simeq K^b(\text{Affine } \mathbb{S}\mathrm{Bim}) \supset K^b(\mathbb{S}\mathrm{Bim})$$

G^u = completion along unipotent element

$\tilde{G}^u = \{(g, F) \in G^u \times G/B \mid g \in F\}$

$St_G = \tilde{G}^u \times_{G^u} \tilde{G}^u$

Main diagram:

$$St_{G/G} \longleftarrow \mathcal{L}^u(G^u/G) \times_{G^u/G} \tilde{G}^u/G \longrightarrow \mathcal{L}^u(G^u/G)$$

Relates DG Hilbert schemes + Markov moves.