# MONODROMY AND KODAIRA FIBRATIONS

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## §1. Motivation

From topology, we know that any finitely presented group is $\pi_1$ of some topological space. In fact, all such groups arise as fundamental groups of four manifolds.

1.1. Question. *what sorts of groups arise as fundamental groups of varieties?*

We'll be restricting to groups associated to certain varieties. Consider a fibration $f\colon S \to B$ with all fibers smooth projective curves, and the base a smooth curve.

1.2. Definition. A fibration $f$ as above is a **Kodaira fibration** if the action of $\pi_1 B$ on the homology of the fibers is nontrivial. (It's not clear that such fibrations exist!)

Now recall that we have the following classical fact: given a fiber bundle $\varphi\colon X \to Y$, $\chi_{top}(X) = \chi_{top}(Y)\chi_{top}(F)$. We can also consider $\sigma$, the intersection form on middle homology. In the case that $\pi_1 Y$ actions trivially on the homology of the fibers, we get an analogous multiplicitavity: $\sigma(X) = \sigma(Y)\sigma(F)$ (due to Chern–Hirzebruch–Serre).

However, Kodaira observed that for a nontrivial action, $\sigma(F) > S$ whereas $\sigma(B) = \sigma(F) = 0$. Motivated by this, Kodaira explicitly constructed a Kodaira fibration as a ramified cover of a product of curves.

The general observation is that the nontrivial action is precisely what breaks the multiplicativity of the signature. It's now natural to ask what kinds of actions are possible.

Consider the action of the fundamental group of the base on the orientation-preserving outer automorphisms of $\pi_1(F)$:

$$\rho\colon \pi_1(B) \to \mathcal{O}^+(\pi_1(F)).$$

Such a homomorphism is equivalent to an extension

$$0 \to \pi_1 F \to \pi_1 S \to \pi_1 B \to 0.$$

So the question then becomes what extensions of this form are possible. This question is hard, and in particular hard to address using algebraic geometry. So we approximate the question by asking about the induced homomorphism

$$\bar{\rho}\colon \pi_1 B \to \mathrm{Gl}(V)$$

where $V = H_1(F, \mathbb{Z}) \otimes \mathbb{Q}$. We might begin by asking about the image of $\bar{\rho}$. Observe that the image factors through $\mathrm{Sp}(V)$. We can further approximate the problem by studying the following:

1.3. Definition. The **connected monodromy group** is the connected component of the identity of the $\mathbb{Q}$-Zariski closure of $\mathrm{Im}(\bar{\rho})$.

Note that the connected monodromy group is a relatively poor approximation of the actual monodromy group – taking a $\mathbb{Q}$-Zariski closure is not a great approximation since it contains a lot of extra stuff. However, this will be our focus.

1.4. Question. *What groups arise as connected monodromy groups of a Kodaira fibration? How do we construct Kodaira fibrations realizing these groups?*

Rather than approaching this in a hands-on way like Kodaira, we'll work abstractly using moduli spaces.

## §2. Connected monodromy groups

To study connected monodromy groups of a Kodaira fibration, we consider

$$J\colon \mathcal{M}_g^{\#} \hookrightarrow \mathcal{A}_g^{\#} = \mathrm{Sl}(2g, \mathbb{Z})\backslash \mathbb{H}_g.$$

Where # denotes the Baily–Borel compactification of $\mathcal{A}_g$ and the closure of $\mathcal{M}_g$ in $\mathcal{A}_g^{\#}$. The map above is given by

$$C \mapsto H_1(C, \mathbb{Z}).$$

(Or by sending a curve to its Jacobian: we're implicitly using an equivalence between the category of polarized abelian varieites and polarized weight one $\mathbb{Z}$-Hodge structures.)

Fact: $\mathcal{M}_g^{\#} \setminus \mathcal{M}_g$ has codimension 2 in $\mathcal{M}_g^{\#}$ if $g \geq 3$. It turns out this implies that, given a general complete intersection of ample divisors in $\mathcal{M}_g$, we get a compact curve lying entirely in $\mathcal{M}_g$. This compact curve induces a Kodaira fibration. This is a completely non-explicit method for building Kodaira fibrations.

Next we might try to tweak this strategy to build Kodaira fibrations with a certain connected monodromy group. Idea: consider a subvariety $Z \subset \mathcal{A}_g$ whose general point has connected monodromy group $T$. If we can show that $Z \setminus \mathcal{M}_g$ has codimension $\geq 2$, then we obtain a Kodaira fibration with connected monodromy $T$ by a similar argument to the above.

2.1. Remark. A summary of previous existence results:

- $\mathcal{M}_2$ is affine, so a Kodaira fibration must have fibers of genus $\geq 3$.
- Zaal gave a by hand constructions for $g = 3$. These explicit examples all have connected monodromy $T = \mathrm{Sp}(6)$ (which is maximal).
- Gonzalez–Diez, Harvey: show no complete curves in $\mathcal{M}_3$ contained in the locus of curves with nontrivial automorphisms.

Nontheless, there is some interesting monodromy in the minimal fiber genus case.

2.2. Theorem (Flapan). *One can show that:*

- *There are five possible connected monodromy groups $T$ of fiber genus 3 Kodaira fibrations.*
- *Three of these are realized by general complete intersection Kodaira fibrations:* $\mathcal{T} = \mathrm{Sp}(6)$, $T = \mathrm{Sp}(4)$, *and* $T = \mathrm{Sl}(2)^3$.
- *It's unknown whether the others occur or not.*