COMPARING COMPACTIFICATIONS OF MODULI SPACES OF PLANE CURVES

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§1. Motivation

What is a moduli space? An algebraic or geometric object whose points correspond to equivalence classes of geometric objects. For example:

- $\overline{\mathcal{M}}_g$, the moduli space of stable genus g curves.
- Moduli spaces of vector bundles on a fixed variety.
- Moduli of hypersurfaces in $\mathbb{P}^n$ (our case will be $n = 2$).

A central question for many moduli spaces: how to compactify?

1.1. Example. Moduli of smooth degree d plane curves $C \subset \mathbb{P}^2$. Given $C \subset P^2$ smooth of degree d, consider

$$\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d))) \simeq \mathbb{P}^{\binom{d+2}{2}-1}.$$

This is a parameter space whose points correspond to degree d curves, which contains a subset U consisting of smooth curves. But, to get a moduli space, we want to quotient by isomorphisms: that is, to consider $\mathcal{U}/\mathbb{P}Gl_3$.

This space is not compact, so we seek to compactify:

1.2. Question. *What "singular fibers" must be added to this moduli space to obtain a compactification (with moduli interprestation)?*

There are a number of ways to address this this:

- GIT. Points in the compactification correspond to orbits of plane curves under the group action (in some sense – there are nuances).

  Pros: easy to write down; points in the boundary are somewhat naturally identified with plane curves, and orbits can be identified using notions like GIT stability.

  Cons: we might get very singular plane curves on the boundary (for d even, you'll get $\frac{d}{2}$ times a conic, which is very non-reduced!); we can't identify the boundary points uniquely with plane curves (this compactification is not modular); the global structure of the moduli space is very singular.
- Alternatively: KSBA (Kollár–Shepherd-Barron–Alexeev) compactification. This takes a more abstract approach. We first enlarge the class of objects to include appropriate singular curves, and get something compact. Hacking's approach in KSBA framework: generalize $(\mathbb{P}^2, C)$ in a different way. Write down a functor to get a moduli

space parameterizing pairs (X, D) where X is a surface and D a divisor such that:

– $(X, (\frac{3}{d}+\epsilon)D)$ is slc (higher dimensional generalization of stable).
– Need a polarization, so we put some condition like $K_X+(\frac{3}{d}+\epsilon)D$ is ample.
– $dK_x + 3D$ linearly equivalent to 0 (note this is satisfied by a plane curve).

The pros: curves are less singular than those in GIT.

The cons: this compactification is very hard to explicitly describe, and the points in the boundary aren't plane curves (they live in more singular surfaces). For example, if $d = 4$ you have points corresponding to curves on $P(1, 1, 4)$ and $P(1, 1, 2) \cup P(1, 1, 2)$ in the boundary.

1.3. Question. *How are (the boundaries of) these two compactifications related?*

We can approach this using a technique called K-stability, which gives a way to parameterize log Fano pairs. A wall-crossing picture allows us to understand what's going on.

First, we need to reinterpret GIT as a moduli of pairs $(\mathbb{P}^2, \epsilon C)$ for $0 < \epsilon << 1$. We can think about the Hacking moduli space $\overline{\mathcal{M}}^H$ as pairs $(\mathbb{P}^2, (\frac{3}{d} + \epsilon)C)$, and from this perspective the Hacking compactification admits a birational map to the GIT compactification, which can be understood as a sequence of blow-ups and flips, etc (known operations). The intermediate moduli spaces $\overline{M}^K_{d,c}$ and the associated wall crossings as the coefficient ranges from ϵ up to $\frac{3}{d} + \epsilon$ can then be studied.

Wall crossings for quartic curves have been described by Hyeon–Lee based on work of Hassett.

DeVleming and her collaborators describe wall crossings for quintic curves, as well as $d = 6$.