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Speaker's Name: Claire Voisin

Talk Title: Stable birational invariants

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STABLE BIRATIONAL INVARIANTS

CLAIRE VOISIN

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1. Rationality

Let X be a smooth projective variety over an algebraically closed field k.

Definition 1.1. We say that X is *unirational* if there exists a dominant rational map $\mathbf{P}_k^N \to X$. (If X is unirational, we can take $N = n := \dim X$.)

Definition 1.2. We say X is *rational* if there exists a birational map $\mathbf{P}_k^n \dashrightarrow X$.
We say X is *stably rational* if $X \times \mathbf{P}_k^r$ is rational for some r.

Evidently, rational $\implies$ stably rational $\implies$ unirational.

The Lüroth problem is concerned with the converse: is a unirational variety rational?

This has a stable version: is a unirational variety stably rational?

Although this is really just a question about the function fields, it turns out to be essential to have a smooth projective model.

For $\dim X \leq 2$, in characteristic 0 the answer has been known since 1970: unirational implies rational.

For $\dim X \geq 3$, the answer is No.

2. Methods of rationality

We will briefly describe the three methods that have been used to address these questions. The first two don't address stable rationality, while the third does.

2.1. The method of Griffiths.

Theorem 2.1 (Griffiths)**.** *A smooth cubic 3-fold $X \hookrightarrow \mathbf{P}_k^4$ is not rational, but it is unirational.*

To see the unirationality, start with a line $L \subset X$ which is tangent to X. Given another line through L, there is a 3rd intersection point with X. This gives a map from the $\mathbf{P}^2$-bundle over L to X.

The stable rationality is still not known. The method used here does not apply to all dimensions.

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2.2. The method of Iskovskih-Manin.

Theorem 2.2 (Iskovskih-Manin). *A smooth quartic 3-fold in $\mathbf{P}^4_k$ is not rational in characteristic* 0.

Although the theorem does not address stable rationality, the criterion used here applies in all dimensions, and was generalized to degree n hypersurfaces in $\mathbf{P}^n$. (They study the birational automorphism group.) However, the method has not been applied to study stable rationality.

2.3. **The method of Artin-Mumford.** This was developed by Artin-Mumford. The idea is to construct stable birational invariants.

3. Stable birational invariants

Definition 3.1. We say X and Y are *stably birational* if there is a birational map $X \times \mathbf{P}^r \dashrightarrow Y \times \mathbf{P}^s$ for some r, s.

This generates an equivalence relation. A stable birational invariant is an invariant on the stable birational equivalence classes.

Example 3.2. For smooth projective X, consider $H^0(\Omega_X^{\otimes \ell})$ for all $\ell \geq 0$. This is a stable birational invariant of X.

Why? The point is that $\Omega_{X \times \mathbf{P}^r} \cong \Omega_X \oplus \Omega_{\mathbf{P}^r}$, hence

$$H^0(\Omega^{\otimes \ell}_{X \times \mathbf{P}^r}) = \bigoplus_{\ell_1 + \ell_2 = \ell} H^0(\Omega_X^{\otimes \ell_1}) \otimes H^0(\Omega_{\mathbf{P}^r}^{\otimes \ell_2}).$$

Then one uses that $\mathbf{P}^r$ has no holomorphic 1-forms.

To finish one has to show the invariance under birational equivalence. Consider the restriction $H^0(\Omega_X^{\otimes \ell}) \to H^0(\Omega_U^{\otimes \ell})$ to an open subset $U \subset X$. This is injective, and it is an isomorphism if the codimension of $X - U$ is at least 2, by normality. Now we are going to use that X and Y are smooth and projective. This implies that any birational $f \colon X \to Y$ is defined on an open subset $U \subset X$ with complement of codimension at least 2. Hence

$$H^0(\Omega_Y^{\otimes \ell}) \hookrightarrow H^0(\Omega_U^{\otimes \ell}) \xleftarrow{\sim} H^0(\Omega_X^{\otimes \ell}).$$

The symmetric argument gives an injection $H^0(\Omega_X^{\otimes \ell}) \hookrightarrow H^0(\Omega_Y^{\otimes \ell})$, and one easily checks that these are inverses of each other.

In characteristic 0, a unirational variety X has $H^0(\Omega_X^{\otimes \ell}) = 0$ for all $\ell > 0$ since the pullback of forms is injective. But in characteristic p, there are counter-examples due to Kollár-Totaro.

4. The Artin-Mumford invariant

Let $X/\mathbf{C}$. We consider the Betti cohomology $H^i_B(X(\mathbf{C}), A)$.

Lemma 4.1. *The "topological Brauer group" $H^3_B(X; \mathbf{Z})_{\mathrm{tors}}$ is a stable birational invariant of smooth, projective X.*

Proof. First we compare X with $X \times \mathbf{P}^r$. By the Kunneth theorem,

$$H^3_B(X \times_k \mathbf{P}^r; \mathbf{Z}) \cong H^3(X;\mathbf{Z}) \oplus H^1(X;\mathbf{Z})$$

and $H^1_B(X;\mathbf{Z})$ has no torsion.

Then we examine birational equivalence. By resolution of singularities, we have a weak factorization of the birational map from X to Y by blowups. Then we just have to show invariance under blowups.

Let $Z \subset X$ and consider the blowup $\widetilde{X}_Z \to X$. We know that

$$H^3_B(\widetilde{X}_Z;\mathbf{Z}) \cong H^3_B(X;\mathbf{Z}) \oplus H^1_B(Z;\mathbf{Z})$$

and $H^1_B(Z;\mathbf{Z})$ is torsion-free, so again the torsion in $H^3(-;\mathbf{Z})$ is preserved. □

Example 4.2. The quartic double solid is a double cover $X \to \mathbf{P}^3$ ramified over a quartic surface. This embeds into L, the total space of $\mathcal{O}(2)$, i.e. Spec $\bigoplus_d \mathcal{O}_{\mathbf{P}^3}(-2d)$. Let $y \in H^0(L, \pi^*\mathcal{O}(2))$.

Artin-Mumford consider $y^2 = f$, the defining equation of X, call it X_f. In the Artin-Mumford example, one chooses a specific f_0 so this has 10 nodes in special position. Let $\widetilde{X}_{f_0}$ be the desingularization. Artin-Mumford showed that $H^3_B(\widetilde{X}_{f_0};\mathbf{Z})_{\mathrm{tors}} \neq 0$. Later we will deform this example to a general quartic double solid X_f, for which the Artin-Mumford example vanishes.

5. Another invariant

Let $X/\mathbf{C}$. Define $Z^4(X)$ to be

$$\left(\frac{H^4_B(X;\mathbf{Z})}{H^4_B(X;\mathbf{Z})_{\mathrm{alg}}}\right)_{\mathrm{tors}}$$

where $H^4_B(X;\mathbf{Z})_{\mathrm{alg}}$ is the subgroup generated by classes of algebraic subvarieties of codimension 2. It is contained in 4-dimensional integral Hodge classes. It is known that

$$\left(\frac{\mathrm{Hdg}^4(X;\mathbf{Z})}{H^4_B(X;\mathbf{Z})_{\mathrm{alg}}}\right)_{\mathrm{tors}} \cong \left(\frac{H^4_B(X;\mathbf{Z})}{H^4_B(X;\mathbf{Z})_{\mathrm{alg}}}\right)_{\mathrm{tors}}$$

According to the Hodge conjecture, $\frac{\mathrm{Hdg}^4(X;\mathbf{Z})}{H^4_B(X;\mathbf{Z})_{\mathrm{alg}}}$ is already torsion. One knows this for unirational varieties.

Lemma 5.1. *The group*

$$\frac{\mathrm{Hdg}^4(X;\mathbf{Z})}{H^4_B(X;\mathbf{Z})_{\mathrm{alg}}}$$

is a stable birational invariant.

Colliot-Thélène has exhibited 6-folds X which are unirational, with $Z^4(X) \neq 0$. Schreieder exhibited 4-folds with this property. Voison proved that $Z^4(X)$ is trivial for 3-folds X if X is unirational or K_X is trivial.

Proof. The point is that when one considers $\mathrm{Hdg}^4(X \times \mathbf{P}^r;\mathbf{Z})$, one finds Hodge classes of X and Hodge classes of degree 2 in X, but the latter are all algebraic by Lefschetz (1,1). □

6. Unramified cohomology

There is a continuous map

$$f\colon X_{an} \to X_{Zar}.$$

This induces a Leray spectral sequence, which is called the Bloch-Ogus spectral sequence. Define $\mathcal{H}^i(A) := R^i f_* A$. This is a sheaf on X_{Zar}, associated to the presheaf

$$U \mapsto H^i_B(U, A).$$

The spectral sequence reads

$$E_2^{pq} = H^p(X_{Zar}, \mathcal{H}^q(A)) \implies H^{p+q}(X_{an}; A).$$

Definition 6.1. We define the *unramified cohomology*

$$H^i_{nr}(X; A) := H^0(X_{Zar}; \mathcal{H}^i(A)).$$

Theorem 6.2. *The unramified cohomology groups are a stable birational invariant.*
If $A = \mathbf{Z}$, then $H^i_{nr}(X; \mathbf{Z}) = 0$ if $i > 0$ and X is unirational.
For X unirational, $Z^4(X) \cong H^3_{nr}(X; \mathbf{Q}/\mathbf{Z})$ and $H^3_B(X; \mathbf{Z})_{\mathrm{tors}} \cong H^2_{nr}(X; \mathbf{Q}/\mathbf{Z})$.

This is highly non-trivial. It uses the Bloch-Kato Conjecture (proved by Voevodsky), which implies that $\mathcal{H}^i(\mathbf{Z})$ are torsion-free.