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Talk Title: Infinity categories and why they are useful, III

Date: 2 / 5 / 19 **Time:** 9 : 30 am / pm **(circle one)**

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INFINITY CATEGORIES AND WHY THEY ARE USEFUL, III

CARLOS SIMPSON

1. Model categories

A 1-category is a category enriched over *sets*: $\mathrm{Map}(x, y)$ is a set.

An ∞-category is a category enriched over *spaces*: $\mathrm{Map}(x, y)$ is a space (simplicial set).

In order to "work with" ∞-categories, one employs the theory of Quillen model categories. In this formalism, one has a 1-category M (example: simplicial sets) and a set of "weak equivalences" $W \subset \mathrm{Mor}(M)$ (example: weak equivalences of simplicial sets), a set of cofibrations and fibrations in $\mathrm{Mor}(M)$. Quillen's theory gives a machine to calculate within the ∞-category $W^{-1}M$.

One can think of a model category as "a choice of coordinates for an ∞-category" (Toën) .

We want to explain how to use the model category framework to get a handle on the homotopy theory of ∞-categories.

2. Model categories of ∞-categories

To work with ∞-categories, we'd like to construct model categories of ∞-categories. Here are some such models:

2.1. Simplicial categories.

- M is the category of simplicial categories.
- W are the Dwyer-Kan equivalences (fully faithful and essentially surjective).
- Cofibrations are "free cell additions".
- Fibrations are determined.

2.2. Segal categories. Recall that Segal categories are bisimplicial sets with discrete A_0 and a homotopy equivalence condition. But a homotopy equivalence condition is not going to be preserved by colimits and limits, which we want to exist in the category of ∞-categories.

So we define a Segal *precategory* to be a bisimplicial set with A_0 discrete.

There is a *Segalification* operation that takes in a Segal precategory A and produces a Segal category $\mathrm{Seg}(A)$.

The "meaning" of this is that a precategory A is really viewed as a system of generators and relations to define an ∞-category $\mathrm{Seg}(A)$.

- M is the category of Segal precategories.

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- Weak equivalences are functors $A \to B$ such that $\mathrm{Seg}(A) \to \mathrm{Seg}(B)$ is a Dwyer-Kan equivalence.
- Cofibrations are injections (meaning levelwise injections).
- Fibrations are determined, although we can say that fibrant objects are Segal categories. We can also say that if A is a Segal category and $A \to A'$ is a fibrant replacement, then $A \to A'$ is a Dwyer-Kan equivalence.

2.3. **Quasicategories.** This theory was introduced by Boardman-Vogt, and developed by Joyal and Lurie.

Quasicategories are simplicial sets that satisfy the "restricted Kan condition". The relation to simplicial categories is as follows. Let A be a simplicial set. We can think of it as a bisimplicial set by making things constant in the second variable (in other words, embed simplicial sets into simplicial spaces), and then apply the Segalification operation. (More directly, one can go directly from Segal precategories to simplicial categories.)

This means that the simplicial set gives a system of generators and relations for a simplicial category. The 0-simplices are objects, the 1-simplices are morphisms, the 2-simplices give "relations" between morphisms, etc.

- M is the category of quasi-categories.
- $X \to Y$ is a weak equivalence if $\mathrm{Seg}(X) \to \mathrm{Seg}(Y)$ is a Dwyer-Kan equivalence.
- Cofibrations are injections.

2.4. **Complete Segal spaces.** This theory was developed by Rezk.

Definition 2.1. If A is an ∞-category, the *interior* ιA is the subcategory with the same objects but only the invertible maps. This is an $(\infty, 1)$-groupoid. So it realizes to a space $|\iota A|$, with $\iota A \xrightarrow{\sim} \pi_\infty(|\iota A|)$.

Remark 2.2. How does $|\iota A|$ compare with $|A|$? $|A|$ is the "hull" of A, which is like the "biggest space containing A". By contrast, $|\iota A|$ is the "smallest space inside A".

Definition 2.3. A *Segal space* is a bisimplicial set A such that

$$A_n \xrightarrow{\sim} A_1 \times_{A_0} \times \ldots \times_{A_0} A_1$$

where A_0 is a space (no longer necessarily discrete), and the fibered products are in a homotopical sense. (It is a good idea to further demand that A is "Reedy fibrant", which means that all the boundary maps are fibrations, hence the fibered product is already derived.)

A *complete Segal space* is a Segal space A such that $A_0 \to |\iota A|$ is an equivalence.

There is a model category structure, but we won't describe it. The advantage of this model is that a map $A \to B$ is an equivalence if and only if it is an equivalence levelwise. For example, A_0 is the *space* of equivalence classes of objects.

3. Comparison of different models

Bergner, Joyal-Kramer, Lurie, etc. show that all of these models are equivalent, by constructing Quillen equivalences between them. Bertrand-Toen gave an axiomatization for the model category of ∞-categories.

4. Operations with ∞-categories

All the techniques of category theory can be done for ∞-categories.

For example, one can form functor categories. If the model structure is *cartesian*, meaning cofibrant objects are preserved by cofibrant pushouts (which is satisfied for Segal categories, quasicategories and complete Segal spaces but *not* simplicial categories), then there is a good functor category. Suppose M has an internal Hom. If A is cofibrant and B is fibrant, then $\underline{\mathrm{Hom}}(A, B) \in M$ is a good mapping object.

We can define an $(\infty, 2)$-category $\infty-\mathrm{CAT}$ (enriched over $(\infty, 1)$-categories) whose objects are fibrant $A \in M$, and given A, B we get an $(\infty, 1)$-category $\underline{\mathrm{Hom}}(A, B)$.

One can form limits, colimits, adjoints, Kan extensions, etc.

One can define stacks, formulate a descent property, etc.

There is a notion of *stable ∞-categories*, which correspond to the ∞-categories enriched over spectra. This includes dg categories, which are then related to A_∞-categories.

5. Application

Let $X \to Y$ be a map of varieties. The rational homotopy types of the fibers X_y for $y \in Y$, form a "local system" over Y. This requires the apparatus of ∞-category to describe properly. It gives an "∞-stack" over Y, which then allows one to make sense of a "non-abelian Gauss-Manin connection". It then fits into a theory of "variation of non-abelian mixed Hodge structures".