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Talk Title: DAG I: the cotangent complex and derived de Rham cohomology

Date: 01 / 31 / 19 **Time:** 9 : 30 [✓ am] / pm **(circle one)**

Please summarize the lecture in 5 or fewer sentences: Introduction to DAG

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DAG I: THE COTANGENT COMPLEX AND DERIVED DE RHAM COHOMOLOGY

BENJAMIN ANTIEAU

1. Motivation

We are going to give an evolutionary diagram.

- In classical algebraic geometry, one considers "algebraic schemes" or "algebraic spaces" such as $\mathbf{P}^n$.
- Then one considers "algebraic stacks" such as the Picard stack $\mathrm{Pic}_{X/k}$ of a curve X/k. This is an "Artin stack".

To make the next step, we need to adopt a new point of view. We can view these objects as sheaves of sets (by the functor of points). We then view sets as "0-truncated spaces" $S_{\leq 0}$, i.e. spaces with non-zero homotopy groups π_i with $i \leq 0$..

More generally, we can consider groupoids, which are "1-truncated spaces" $S_{\leq 1}$.

If we continue, we enter the realm of "higher stacks", e.g. $K(\mathbf{G}_m, n)$. The isotopy of this space is $K(\mathbf{G}_m, n-1)$.

Example 1.1. We can reformulate $\mathrm{Pic}_{X/k}$ as the mapping stack from X to $K(\mathbf{G}_m, 1)$, which also goes by the name $B\mathbf{G}_m = [\mathrm{pt}/\mathbf{G}_m]$. We can then view $K(\mathbf{G}_m, 2) = [\mathrm{pt}/B\mathbf{G}_m]$.

Example 1.2. What does it mean to give a map from a scheme to $K(\mathbf{G}_m, n)$? By definition $\mathrm{Map}(X, K(\mathbf{G}_m, n))$ is a topological space, with

$$\pi_i \, \mathrm{Map}(X, K(\mathbf{G}_m, n)) \cong \begin{cases} H^{n-i}_{\text{ét}}(X, \mathbf{G}_m) & 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We have now expanded our world of "geometric objects" to include sheaves of groupoids, or sheaves of n-truncated spaces. The fundamental idea of derived algebraic geometry is to allow the sheaf of functions itself to be a sheaf of topological spaces. This leads to the notion of "derived schemes".

Example 1.3. An example of an affine derived scheme is $\mathrm{Spec}\,(k \otimes^L_{k[x]} k)$, where the maps $k[x] \to k$ send $x \mapsto 0$.

2. Simplicial commutative rings

We are now going to introduce a model for what should be the "commutative rings of derived algebraic geometry".

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Consider the derived category $D(\mathbf{Z})_{\geq 0}$, where we are considering only the complexes C_* such that $H_i(C_*) = 0$ for $i < 0$ (this condition is also called "connective"). This can be enhanced to a symmetric monoidal category using the derived tensor product $\otimes^L$.

What should we take as the commutative algebras in $(D(\mathbf{Z})_{\geq 0}, \otimes^L)$? There are three different answers:

- E_∞-ring spectra,
- simplicial commutative rings,
- Over $\mathbf{Q}$, one can work with $\mathbf{Q}$-CDGAs.

These are not equivalent in general, although they are equivalent rationally. We will work with the second.

We introduce the *simplex category* Δ, which is the category of non-empty finite ordered sets, with morphisms being order-preserving maps of sets.

Example 2.1. We write [0] for $\{0\}$, [1] for $\{0, 1\}$, and [2] for $\{0, 1, 2\}$. Evidently we have two maps $[0] \to [1]$ and one map $[1] \to 0$, etc.

Definition 2.2. Let $\mathcal{C}$ be a category. We let $s\mathcal{C} = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ be the category of *simplicial objects in* $\mathcal{C}$.

Example 2.3. There is an "equivalence" between sSets and the category of topological spaces, at the level of *homotopy categories*. This means we specify a notion of "weak equivalence" on each side (in topological spaces it is the usual notion, wherein maps inducing isomorphisms of π_* are weak equivalences), and the "localization" of each side with respect to these are equivalent.

Let Δ^n be the pre-sheaf $\mathrm{Hom}_\Delta(-, [n])$ i.e. the Yoneda embedding of $[n]$. Let Δ^n_{top} be the usual n-simplex. This induces a functor from sSets to topological spaces, called *geometric realization*, giving by presenting a simpicial set as a colimit of the representable objects Δ^n and taking that colimit in Top. We then pull back the notion of weak equivalence along this functor.

We can view Δ^*_{top} as a cosimplicial object in Top. Hence $\mathrm{Hom}_{\mathrm{Top}}(\Delta^*_{\mathrm{top}}, X)$ is a simplicial object in Top. This is the right adjoint to geometric realization.

Let's make a connection to something familiar. Given a topological space X, we make a simplicial set $\mathrm{Sing}(X) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^*_{\mathrm{top}}, X)$. Then we form a simplicial abelian group $\mathbf{Z}[\mathrm{Sing}(X)]$. We can extract from this a chain complex $C_*(\mathbf{Z}[\mathrm{Sing}(X)])$, and by definition

$$H_i(C_*(\mathbf{Z}[\mathrm{Sing}(X)])) \cong H_i^{\mathrm{sing}}(X; \mathbf{Z}).$$

Example 2.4. The *Dold-Kan correspondence* furnishes an equivalence $s\mathrm{Ab} \cong D(\mathbf{Z})_{\geq 0}$. Given a simplicial abelian group

$$M_0 \leftleftarrows M_1 \dots$$

we make a chain complex with differentials $\sum (-1)^i d_i$:

$$M_0 \xleftarrow{d_0 - d_1} M_1 \leftarrow \dots$$

Example 2.5. The category of *simplicial commutative rings* is

$$s\mathrm{CAlg}_k = \mathrm{Fun}(\Delta^{op}, \mathrm{CAlg}_k).$$

There is a notion of weak equivalence, which is weak equivalence of the underlying simplicial sets. But moreover, there is a *model category structure*, which specifies weak equivalences but also specifies a "right way" to perform certain derived operations. For this reason, this is sometimes called a "non-abelian derived category".

For $R \in s\mathrm{CAlg}$, $\pi_* R$ has the structure of a *graded commutative ring.* This means that

$$xy = (-1)^{|x||y|} yx$$

and $x^2 = 0$ if $|x|$ is odd.

There is an adjunction Sets → CAlg_k sending $S \mapsto k[S]$. Given $R \in \mathrm{CAlg}_k$, we can make a simplicial k-algebra

$$S_\bullet : \ldots k[k[R]] \rightrightarrows k[R]$$

which is the analogue of a "projective resolution" for commuative rings. Let $\mathrm{CAlg}_k^{\mathrm{poly}}$ be the category of finitely generated polynomial k-algebras. There is a fully faithful embedding to $\mathrm{Ind}(\mathrm{CAlg}_k^{\mathrm{poly}})$, which is the category obtained by formally adjoining colimits, and this includes into $s\mathrm{CAlg}_k$.

Given a functor $F\colon \mathrm{CAlg}_k^{\mathrm{poly}} \to \mathcal{C}$, there is a way to extend to $LF\colon s\mathrm{CAlg}_k \to \mathcal{C}$. The recipe is as follows. Given $R \in \mathrm{CAlg}_k$, make $S_* \xrightarrow{\sim} R$ as above. Then we have $F(S_*) \in s\mathcal{C}$, and define

$$LF(R) = \varinjlim_{\Delta} F(S_*) =: |F(S_*)|.$$

3. The cotangent complex

We give several constructions.

(a) Given $k \to R$, we have explained that we can make a simplicial a "good" simplicial "resolution" $S_\bullet \xrightarrow{\sim} R$. Then we define

$$L_{R/k} := \Omega^1_{S_\bullet/k} \otimes_{S_\bullet} R \in s\mathrm{Mod}_R \cong D(R)_{\geq 0}.$$

(b) We have a functor $\Omega^1_{-/k}\colon \mathrm{CAlg}_k^{\mathrm{poly}} \to D(k)_{\geq 0}$. We can then extend this to a $L\Omega^1_{-/k}$ on $s\mathrm{CAlg}_k$ on as discussed previously. However, there are several deficiencies, e.g. we don't see the R-module structure.

(c) We'll correct the previous issues by phrasing a universal property. Let $k \to R \to S$ and M be an S-module in $D(S)_{\geq 0}$. We will define $L_{R/k}$ to have the universal property that (the topological space)

$$\mathrm{Map}_R(L_{R/k}, M) \cong \mathrm{Map}_{s\mathrm{CAlg}_{k/S}}(R, S \oplus M).$$

Here $\mathrm{Map}_{s\mathrm{CAlg}_{k/S}}$ is the slice category of simplicial commutative rings equipped with a map to S, and $S \oplus M$ is the square-zero extension of S by M. Note that this makes sense for $M \in D(S)_{\geq 0}$!

Exercise 3.1. Show that $\pi_0 L_{R/k} \cong \Omega^1_{R/k}$.

Exercise 3.2. Show that $S \otimes^L_R L_{R/k} \to L_{S/k} \to L_{S/R}$ is an exact triangle.

Example 3.3. Show that $T \otimes_k L_{R/k} \cong L_{R \otimes^L_k T/T}$

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & R \arrow[d] \\
T \arrow[r] & R \otimes^L_k T
\end{tikzcd}$$

Exercise 3.4. Let R be a perfect algebra over $\mathbf{F}_p$. Using that the Frobenius morphism $\varphi\colon R \to R$, which sends $x \mapsto x^p$, is an isomorphism, show that $L_{R/\mathbf{F}_p} \cong 0$.

4. Derived de Rham cohomology

There is a functor

$$\mathrm{dR}_{R/k}\colon \mathrm{CAlg}_k^{\mathrm{poly}} \to D(k)$$

sending R to the chain complex $(R \mapsto \Omega^1_{R/k} \to \Omega^2_{R/k} \to \ldots)$. We can then "derive" this functor to get a "derived de Rham cohomology" functor

$$L\,\mathrm{dR}_{R/k}\colon s\mathrm{CAlg}_k^{\mathrm{poly}} \to D(k).$$

There is something to be cautious about. If k is a $\mathbf{Q}$-algebra and $R \in \mathrm{CAlg}_k^{\mathrm{poly}}$, then $\mathrm{dR}_{R/k} \cong k$. This implies that $L\,\mathrm{dR}_{S/k} \cong k$ for any $s \in s\mathrm{CAlg}_k$, and this isn't very interesting.

A fix was given by Bhargav Bhatt, which gives the answer you "want". We also have a Hodge filtration F^*_H on $\mathrm{dR}_{R/k}$, with

$$\mathrm{gr}^i_H(\mathrm{dR}_{R/k}) \cong \Omega^i_{R/k}[-i].$$

We then try to take the derived functor remembering the filtration. We then get a filtration $F^*_H L\,\mathrm{dR}_{-/k}$ such that

$$\mathrm{gr}^i_H\, L\,\mathrm{dR}_{-/k} \cong L(\wedge^i L_{-/k}[-i]).$$

This filtration isn't complete, so we define

$$\widehat{L\,\mathrm{dR}}_{-/k} := \lim_i \frac{L\,\mathrm{dR}_{-/k}}{F^i_H}$$

Theorem 4.1 (Bhatt, Grothendieck, Hartshorne)**.** *Let $X/\mathbf{C}$ be finite type. Then*

$$R\Gamma(X, \widehat{L\,\mathrm{dR}}_{\mathcal{O}_X/k}) \cong R\Gamma_{\mathrm{sing}}(X(\mathbf{C}); \mathbf{C}).$$